Multiple conjugation (bi)quandle colorings for handlebody-knots and their applications

著者	Murao Tomo
year	2018
その他のタイトル	ハンドル体結び目の多重共役（バイ）カンドル彩色とその応用
学位授与大学	筑波大学 (University of Tsukuba)
学位授与年度	2018
報告番号	12102甲第8787号
URL	http://doi.org/10.15068/00153838

Multiple conjugation (bi)quandle colorings for handlebody-knots and their applications

Tomo Murao

June 2018

Multiple conjugation (bi)quandle colorings for handlebody-knots and their applications

Tomo Murao
Doctoral Program in Mathematics

Submitted to the Graduate School of
Pure and Applied Sciences
in Partial Fulfillment of the Requirements
for the Degree of Doctor of Philosophy in
Science

at the
University of Tsukuba

Abstract

This thesis consists of four main results.

First, we give a formula of the connected component decomposition of the Alexander quandle: $\mathbb{Z}[t^{\pm 1}]/(f_1(t), \ldots, f_k(t)) = \bigsqcup_{i=0}^{a-1} \mathrm{Orb}(i)$, where $a = \gcd(f_1(1), \ldots, f_k(1))$. We show that each connected component $\mathrm{Orb}(i)$ is isomorphic to $\mathbb{Z}[t^{\pm 1}]/J$ with an explicit ideal J. We introduce a decomposition of a quandle into the disjoint union of maximal connected subquandles. In particular, it is obtained by iterating a connected component decomposition when the quandle is finite.

Secondly, we give lower bounds for the Gordian distance and the unknotting number of handlebody-knots by using Alexander biquandle colorings. We construct handlebody-knots with arbitrary Gordian distance and unknotting number.

Thirdly, we give a lower bound for the tunnel number of handlebody-knots. We also give a lower bound for the cutting number, which is a "dual" notion to the tunnel number in handlebody-knot theory. We provide necessary conditions to be constituent handlebody-knots by using G-family of quandles colorings. The above two evaluations are obtained as the corollaries. Furthermore, we construct handlebody-knots with arbitrary tunnel number and cutting number.

Finally, we define a functor $\mathcal{Q}$ from the category of multiple conjugation biquandles to that of multiple conjugation quandles. We show that for any multiple conjugation biquandle X, there is a one-to-one correspondence between the set of X-colorings and that of $\mathcal{Q}(X)$-colorings diagrammatically for any handlebody-link and spatial trivalent graph. In particular, we prove that the set of G-family of Alexander biquandles colorings is isomorphic to that of G-family of Alexander quandles colorings as modules.

Contents

Chapter 1

Introduction

A handlebody-knot [14] is a handlebody embedded in the 3-sphere S^3. Two handlebody-knots are equivalent if there exists an orientation-preserving self-homeomorphism of S^3 sending one to the other, that is, they are transformed into each other by an isotopy of S^3. Handlebody-knot theory is a generalization of knot theory since a classical knot corresponds to a genus 1 handlebody-knot by taking a regular neighborhood. We say that a handlebody-knot is trivial when its exterior is a handlebody. We define a diagram of a handlebody-knot by a diagram of a spatial trivalent graph which is a spine of the handlebody-knot. The equivalence class of handlebody-knots is completely described by six local deformations of their diagrams, called Reidemeister moves [14].

The main purpose of handlebody-knot theory is to classify handlebody-knots up to the equivalence relation and to characterize properties of each handlebody-knot. Ishii, Kishimoto, Moriuchi and Suzuki [21] gave a table of genus two handlebody-knots up to six crossings, and classified them according to the crossing number and the irreducibility. They were proved to be mutually distinct by using the fundamental groups of their exteriors, quandle cocycle invariants in [17] and some topological arguments in [22, 32]. Some invariants of classical knots have been modified and generalized to construct invariants of handlebody-knots. Unfortunately, however, invariants of handlebody-knots derived from topologies of their exteriors do not work well unlike that of classical knots since classical knots are completely determined by their exteriors[11], but handlebody-knots are not [22, 32, 39]. On the other hand, there are also invariants of handlebody-knots derived from the Reidemeister moves. Ishii, Iwakiri, Jang and Oshiro distinguished handlebody-knots with homeomorphic exteriors by using G-family of quandles [18], which is an algebraic system derived from the Reidemeister moves. Furthermore, these G-families were generalized to an algebraic system called a multiple conjugation quandle in [15]. In this thesis, we study some geometric properties of handlebody-knots, which few studies have focused on, through discussions on colorings by these algebraic systems.

A quandle [27, 34] is a universal algebraic system derived from Reidemeister moves for oriented classical knots to define an arc coloring invariant, which is a map from the set of arcs of a knot diagram to a quandle satisfying some conditions obtained from each crossing of the diagram. The fundamental quandle, defined in [27], is a complete invariant of classical knots, although it is difficult to show that two qundles are isomorphic in general. Hence we often classify classical knots by considering a quanlde representation of the fundamental quandle, called a quandle coloring. An arc coloring is a diagrammatic definition of a quandle coloring.

A multiple conjugation quandle [15] is a universal algebraic system to define arc coloring invariants for handlebody-knots. The set of multiple conjugation quandle colorings of a handlebody-knot diagram becomes a vector space for some multiple conjugation quandles. In this thesis, by considering the effects of an operation for handlebody-knots on the vector space, we give lower bounds for some geometric invariants of handlebody-knots.

A biquandle [9, 10, 30], which is a generalization of a quandle, is a universal algebraic system to define a semi-arc coloring invariant for oriented classical knots. A multiple conjugation biquandle is introduced in [19] as a biquandle version of a multiple conjugation quandle. Unfortunately, it is known that there is a one-to-one correspondence between the set of biquandle colorings and that of quandle colorings for any classical knots [24, 25, 48]. On the other hand, it has not been known whether an invariant obtained from multiple conjugation biquandle colorings is more effective than one obtained from multiple conjugation quandle colorings.

This thesis is organized as follows. In Chapter 2, we review the definitions of handlebody-knots, spatial trivalent graphs, their Reidemeister moves and so on. In Chapter 3, we introduce the definitions of a (bi)quandle, a multiple conjugation (bi)quandle, a G-family of (bi)quandles and colorings for handlebody-knots by using them. In Chapter 4, we show the uniqueness of the maximal connected subquandle decomposition of a quandle and provide how to obtain the decomposition. Moreover, we determine the decomposition concretely for some Alexander quandles. In Chapter 5, we give lower bounds for the Gordian distance and the unknotting number of handlebody-knots and construct handlebody-knots with arbitrary Gordian distance and unknotting number. In Chapter 6, we provide necessary conditions to be constituent handlebody-knots. Furthermore, we give lower bounds for the tunnel number and the cutting number of handlebody-knots and construct handlebody-knots with arbitrary tunnel number and cutting number. In Chapter 7, we show that there is a one-to-one correspondence between the set of multiple conjugation quandle colorings and that of multiple conjugation biquandle colorings diagrammatically for any handlebody-knot and spatial trivalent graph.

Chapter 2

Preliminaries

2.1 Handlebody-knots

A *handlebody-link* is the disjoint union of handlebodies embedded in the 3-sphere S^3 [14]. A *handlebody-knot* is a one component handlebody-link, which is a generalization of a knot with respect to a genus. In this thesis, we assume that every component of a handlebody-link is of genus at least 1. An S^1*-orientation* of a handlebody-link is an orientation of all genus 1 components of the handlebody-link, where an orientation of a solid torus is an orientation of its core S^1. Two S^1-oriented handlebody-links H_1 and H_2 are *equivalent*, denoted $H_1 \cong H_2$, if there exists an orientation-preserving self-homeomorphism of S^3 sending one to the other preserving the S^1-orientation, that is, they are transformed into each other by an isotopy of S^3 preserving the S^1-orientation.

A *spatial trivalent graph* is a finite trivalent graph embedded in S^3. In this thesis, a trivalent graph may have a circle component, which has no vertices. A *Y-orientation* of a spatial trivalent graph is an orientation of the graph without sources and sinks with respect to the orientation (Figure 2.1). A vertex of a Y-oriented spatial trivalent graph can be allocated a sign; the vertex is said to be positive or negative, or to have sign $+1$ or -1. The standard convention is shown in Figure 2.1. For a Y-oriented spatial trivalent graph K and an S^1-oriented handlebody-link H, we say that K *represents* H if H is a regular neighborhood of K and the S^1-orientation of H agrees with the Y-orientation. Any S^1-oriented handlebody-link can be represented by some Y-oriented spatial trivalent graph. We define a *diagram* of an S^1-oriented handlebody-link by a diagram of a Y-oriented spatial trivalent graph representing the handlebody-link. An S^1-oriented handlebody-link is *trivial* if it has a diagram with no crossings. In particular, H is an S^1-oriented trivial handlebody-knot if and only if its exterior is a handlebody.

Then the following theorem holds.

Theorem 2.1.1 ([16]). *Let D_1 and D_2 be diagrams of S^1-oriented handlebody-links H_1 and H_2 respectively. Then H_1 and H_2 are equivalent if and only if D_1 and D_2 are related by a finite sequence of R1–R6 moves depicted in Figure 2.2 preserving Y-orientations, called the Reidemeister moves.*

We note that the R1–R5 moves in Figure 2.2 are the Reidemeister moves for

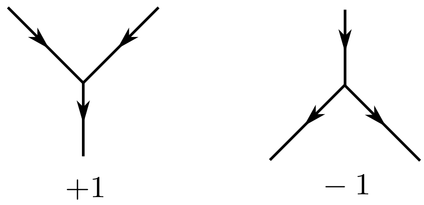


Figure 2.1: Y-orientations and signs of a vertex.

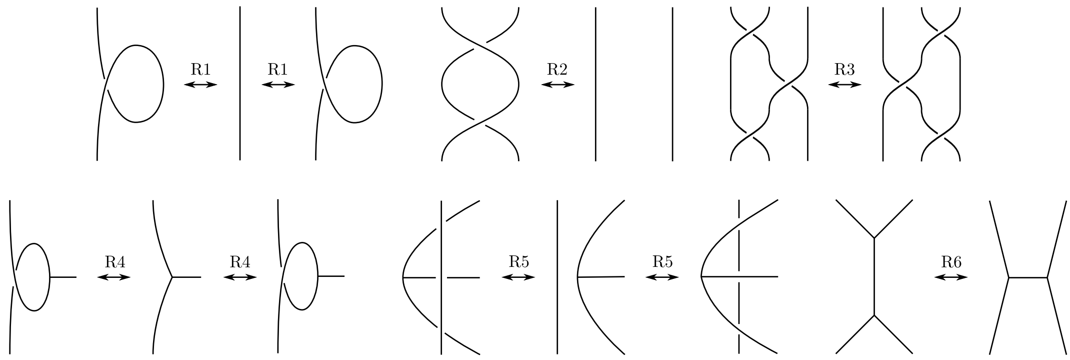


Figure 2.2: The Reidemeister moves for handlebody-links.

spatial trivalent graphs [28, 49, 51]. Hence we can regard handlebody-links as a quotient structure of spatial trivalent graphs.

2.2 Notations

Throughout the thesis, for any diagram D of an S^1-oriented handlebody-link, we denote by $\mathcal{A}(D)$, $\mathcal{SA}(D)$, $C(D)$ and $V(D)$ the set of all arcs, semi-arcs, crossings and vertices of D respectively, where a semi-arc is a piece of a curve each of whose endpoints is a crossing or a vertex. An orientation of an arc of D is also represented by the normal orientation obtained by rotating the usual orientation counterclockwise by $\pi/2$ on the diagram. For any $m \in \mathbb{Z}_{\geq 0}$, we put $\mathbb{Z}_m := \mathbb{Z}/m\mathbb{Z}$. For any set X, we denote by $\#X$ or $|X|$ the cardinality of X.

In this thesis, we often omit brackets. When we omit brackets, we apply binary operations from left on expressions, except for group multiplications, which we always apply first. For example, we write $a *_1 b *_2 cd *_3 (e *_4 f *_5 g)$ for $((a *_1 b) *_2 (cd)) *_3 ((e *_4 f) *_5 g)$ simply, where each $*_i$ is a binary operation, and c and d are elements of the same group.

Chapter 3

Multiple conjugation (bi)quandle colorings for handlebody-knots

3.1 Quandles and biquandles

A quandle is an algebraic system whose axioms are derived from the Reidemeister moves for oriented links, and a biquandle is a generalization of a quandle. In this section, we recall the definitions of a quandle and a biquandle.

Definition 3.1.1 ([27, 34])**.** A *quandle* is a non-empty set X with a binary operation $* : X \times X \to X$ satisfying the following axioms.

- For any $x \in X$, $x * x = x$.
- For any $y \in X$, the map $S_y : X \to X$ defined by $S_y(x) = x * y$ is a bijection.
- For any $x, y, z \in X$, $(x * y) * z = (x * z) * (y * z)$.

We define the *type* of a quandle X, denoted $\operatorname{type} X$, by

$$\operatorname{type} X = \min\{n \in \mathbb{Z}_{>0} \mid a *^n b = a \text{ (for any } a, b \in X)\},$$

where we set $a *^i b := S_b^i(a)$ and $\min \emptyset := \infty$ for any $i \in \mathbb{Z}$, $a, b \in X$ and the empty set $\emptyset$. We note that $(X, *^i)$ is also a quandle for any $i \in \mathbb{Z}$, and any finite quandle is of finite type.

We give some examples of quandles. Let G be a group. We define a binary operation $* : G \times G \to G$ by $a * b = b^{-1}ab$ for any $a, b \in G$. Then G is a quandle, which is called a *conjugation quandle* and denoted by $\mathrm{Conj}(G)$. The second example is a *dihedral quandle* $R_m := \mathbb{Z}_m$ for any $m \in \mathbb{Z}_{\geq 0}$. We define a binary operation $* : R_m \times R_m \to R_m$ by $a * b = 2b - a$ for any $a, b \in R_m$. Then R_m is a quandle. The third example is obtained from an $R[t^{\pm 1}]$-module X, where R is a commutative ring. For any $a, b \in X$, we define $a * b = ta + (1 - t)b$. Then X is a quandle, called an *Alexander quandle*.

Let $(X, *_X)$ and $(Y, *_Y)$ be quandles. A *homomorphism* $\phi : X \to Y$ is a map from X to Y satisfying $\phi(x *_X y) = \phi(x) *_Y \phi(y)$ for any $x, y \in X$. We call a bijective homomorphism an *isomorphism*. X and Y are *isomorphic*, denoted $X \cong Y$, if there exists an isomorphism from X to Y. We call an isomorphism from X to X an *automorphism* of X. For any $a \in X$ and $n \in \mathbb{Z}$, the map $S_a^n : X \to X$ is an automorphism of X.

Definition 3.1.2 ([9, 10, 30])**.** A *biquandle* is a non-empty set X with binary operations $\overline{*}, \underline{*} : X \times X \to X$ satisfying the following axioms.

- For any $x \in X$, $x \,\underline{*}\, x = x \,\overline{*}\, x$.
- For any $y \in X$, the map $\underline{S}_y : X \to X$ defined by $\underline{S}_y(x) = x \,\underline{*}\, y$ is a bijection.

  For any $y \in X$, the map $\overline{S}_y : X \to X$ defined by $\overline{S}_y(x) = x \,\overline{*}\, y$ is a bijection.

  The map $S : X \times X \to X \times X$ defined by $S(x, y) = (y \,\overline{*}\, x, x \,\underline{*}\, y)$ is a bijection.
- For any $x, y, z \in X$,

$$\begin{aligned}(x \,\underline{*}\, y) \,\underline{*}\, (z \,\underline{*}\, y) &= (x \,\underline{*}\, z) \,\underline{*}\, (y \,\overline{*}\, z),\\ (x \,\underline{*}\, y) \,\overline{*}\, (z \,\underline{*}\, y) &= (x \,\overline{*}\, z) \,\underline{*}\, (y \,\overline{*}\, z),\\ (x \,\overline{*}\, y) \,\overline{*}\, (z \,\overline{*}\, y) &= (x \,\overline{*}\, z) \,\overline{*}\, (y \,\underline{*}\, z).\end{aligned}$$

We note that $(X, *)$ is a quandle if and only if $(X, *, \overline{*})$ is a biquandle with $x \overline{*} y = x$. Let $(X, \underline{*}, \overline{*})$ be a biquandle. For any $i \in \mathbb{Z}$ and $a, b \in X$, we define $a \,\underline{*}^i\, b := \underline{S}_b^i(a)$ and $a \,\overline{*}^i\, b := \overline{S}_b^i(a)$. Then we define two families of binary operations $\underline{*}^{[n]}, \overline{*}^{[n]} : X \times X \to X (n \in \mathbb{Z})$ by the equalities:

$$\begin{aligned}&a \,\underline{*}^{[0]}\, b = a, \quad a \,\underline{*}^{[1]}\, b = a \,\underline{*}\, b, \quad a \,\underline{*}^{[i+j]}\, b = (a \,\underline{*}^{[i]}\, b) \,\underline{*}^{[j]}\, (b \,\underline{*}^{[i]}\, b),\\ &a \,\overline{*}^{[0]}\, b = a, \quad a \,\overline{*}^{[1]}\, b = a \,\overline{*}\, b, \quad a \,\overline{*}^{[i+j]}\, b = (a \,\overline{*}^{[i]}\, b) \,\overline{*}^{[j]}\, (b \,\overline{*}^{[i]}\, b)\end{aligned}$$

for any $i, j \in \mathbb{Z}$ [19, 23]. Since $a = a \underline{*}^{[0]} b = (a \underline{*}^{[-1]} b) \underline{*}^{[1]} (b \underline{*}^{[-1]} b) = (a \underline{*}^{[-1]} b) \underline{*} (b \underline{*}^{[-1]} b)$, we have $a \,\underline{*}^{[-1]}\, b = a \,\underline{*}^{-1}\, (b \,\underline{*}^{[-1]}\, b)$ and $(b \,\underline{*}^{[-1]}\, b) \,\underline{*}\, (b \,\underline{*}^{[-1]}\, b) = b$, where we note that $b \,\underline{*}^{[-1]}\, b$ is the unique element satisfying $(b \,\underline{*}^{[-1]}\, b) \,\underline{*}\, (b \,\underline{*}^{[-1]}\, b) = b$ [19].

We define the *type* of a biquandle X, denoted $\operatorname{type} X$, by

$$\operatorname{type} X = \min\{n \in \mathbb{Z}_{>0} \mid a \,\underline{*}^{[n]}\, b = a = a \,\overline{*}^{[n]}\, b \text{ (for any } a, b \in X)\},$$

where we remind that $\min \emptyset = \infty$ for the empty set $\emptyset$. Any finite biquandle is of finite type [23].

Let X be an $R[s^{\pm 1}, t^{\pm 1}]$-module, where R is a commutative ring. For any $a, b \in X$, we define $a \underline{*} b = ta + (s - t)b$ and $a \overline{*} b = sa$. Then X is a biquandle, called an *Alexander biquandle*. Any Alexander biquandle with $s = 1$ coincides with an Alexander quandle. For an Alexander biquandle X, we have $a \,\underline{*}^{[n]}\, b = t^n a + (s^n - t^n)b$ and $a \,\overline{*}^{[n]}\, b = s^n a$ for any $a, b \in X$.

3.2 Multiple conjugation quandles and multiple conjugation biquandles

A multiple conjugation quandle (MCQ) is introduced as the universal symmetric quandle with a partial multiplication to define coloring invariants for handlebody-links, where a partial multiplication is an operation used at trivalent vertices. A multiple conjugation biquandle (MCB) is a biquandle version of an MCQ. In this section, we recall the definitions of an MCQ and an MCB.

Definition 3.2.1 ([15])**.** A *multiple conjugation quandle (MCQ)* X is the disjoint union of groups $G_\lambda (\lambda \in \Lambda)$ with a binary operation $* : X \times X \to X$ satisfying the following axioms.

- For any $a, b \in G_\lambda$, $a * b = b^{-1}ab$.
- For any $x \in X$ and $a, b \in G_\lambda$, $x * e_\lambda = x$ and $x * ab = (x * a) * b$, where e_λ is the identity of G_λ.
- For any $x, y, z \in X$, $(x * y) * z = (x * z) * (y * z)$.
- For any $x \in X$ and $a, b \in G_\lambda$, $ab * x = (a * x)(b * x)$, where $a * x, b * x \in G_\mu$ for some $\mu \in \Lambda$.

We remark that an MCQ itself is a quandle. Let $X = \bigsqcup_{\lambda \in \Lambda} G_\lambda$ and $Y = \bigsqcup_{\mu \in M} G_\mu$ be MCQs. An *MCQ homomorphism* $\phi : X \to Y$ is a map from X to Y satisfying $\phi(x * y) = \phi(x) * \phi(y)$ for any $x, y \in X$ and $\phi(ab) = \phi(a)\phi(b)$ for any $\lambda \in \Lambda$ and $a, b \in G_\lambda$. We call a bijective MCQ homomorphism an *MCQ isomorphism*. X and Y are *isomorphic* if there exists an MCQ isomorphism from X to Y.

Next, we review the definition of a multiple conjugation biquandle (MCB). Let X be the disjoint union of groups $G_\lambda (\lambda \in \Lambda)$. We denote by G_a the group G_λ containing $a \in X$. We also denote by e_λ the identity of G_λ. Then the identity of G_a is denoted by e_a for any $a \in X$.

Definition 3.2.2 ([19])**.** A *multiple conjugation biquandle (MCB)* X is the disjoint union of groups $G_\lambda (\lambda \in \Lambda)$ with binary operations $\underline{*}, \overline{*} : X \times X \to X$ satisfying the following axioms.

- For any $x, y, z \in X$,
$$(x \,\underline{*}\, y) \,\underline{*}\, (z \,\underline{*}\, y) = (x \,\underline{*}\, z) \,\underline{*}\, (y \,\overline{*}\, z),$$
$$(x \,\underline{*}\, y) \,\overline{*}\, (z \,\underline{*}\, y) = (x \,\overline{*}\, z) \,\underline{*}\, (y \,\overline{*}\, z),$$
$$(x \,\overline{*}\, y) \,\overline{*}\, (z \,\overline{*}\, y) = (x \,\overline{*}\, z) \,\overline{*}\, (y \,\underline{*}\, z).$$
- For any $a, x \in X$, the maps $\underline{*}x : G_a \to G_{a \underline{*} x}$ and $\overline{*}x : G_a \to G_{a \overline{*} x}$ are group homomorphisms.
- For any $x \in X$ and $a, b \in G_\lambda$,
$$x \,\underline{*}\, ab = (x \,\underline{*}\, a) \,\underline{*}\, (b \,\overline{*}\, a), \qquad x \,\underline{*}\, e_\lambda = x,$$
$$x \,\overline{*}\, ab = (x \,\overline{*}\, a) \,\overline{*}\, (b \,\overline{*}\, a), \qquad x \,\overline{*}\, e_\lambda = x,$$
$$a^{-1}b \,\overline{*}\, a = ba^{-1} \,\underline{*}\, a.$$

We remark that an MCB itself is a biquandle. Let $X = \bigsqcup_{\lambda\in\Lambda} G_\lambda$ and $Y = \bigsqcup_{\mu\in M} G_\mu$ be MCBs. An *MCB homomorphism* $\phi : X \to Y$ is a map from X to Y satisfying $\phi(x \,\underline{*}\, y) = \phi(x) \,\underline{*}\, \phi(y)$ and $\phi(x \,\overline{*}\, y) = \phi(x) \,\overline{*}\, \phi(y)$ for any $x, y \in X$ and $\phi(ab) = \phi(a)\phi(b)$ for any $\lambda \in \Lambda$ and $a, b \in G_\lambda$. We call a bijective MCB homomorphism an *MCB isomorphism*. X and Y are *isomorphic* if there exists an MCB isomorphism from X to Y.

3.3 G-families of quandles and G-families of biquandles

A G-family of quandles is an algebraic system whose axioms are motivated from handlebody-knot theory and yields an MCQ. A G-family of biquandles is a biquandle version of a G-family of quandles and yields an MCB. In this section, we recall the definitions of a G-family of quandles and a G-family of biquandles.

Definition 3.3.1 ([18])**.** Let G be a group with identity element e. A G-family of quandles is a non-empty set X with a family of binary operations $*^g : X \times X \to X$ $(g \in G)$ satisfying the following axioms.

- For any $x \in X$ and $g \in G$, $x *^g x = x$.
- For any $x, y \in X$ and $g, h \in G$, $x *^{gh} y = (x *^g y) *^h y$ and $x *^e y = x$.
- For any $x, y, z \in X$ and $g, h \in G$, $(x *^g y) *^h z = (x *^h z) *^{h^{-1}gh} (y *^h z)$.

Let R be a ring and let G be a group with identity element e. Let X be a right $R[G]$-module, where $R[G]$ is the group ring of G over R. Then $(X, \{*^g\}_{g\in G})$ is a G-family of quandles, called a *G-family of Alexander quandles*, with $x *^g y = xg + y(e - g)$ [18]. Let $(X, *)$ be a quandle and let m be the type of X. Then $(X, \{*^i\}_{i\in\mathbb{Z}_{km}})$ is a $\mathbb{Z}_{km}$-family of quandles for any $k \in \mathbb{Z}_{\geq 0}$ [18]. In particular, when X is an Alexander quandle, $(X, \{*^i\}_{i\in\mathbb{Z}_{km}})$ is called a *$\mathbb{Z}_{km}$-family of Alexander quandles.*

Let $(X, \{*^g\}_{g\in G})$ be a G-family of quandles. Then $X \times G = \bigsqcup_{x\in X}\{x\} \times G$ is an MCQ with

$$(x, g) * (y, h) := (x *^h y, h^{-1}gh), \qquad (x, g)(x, h) := (x, gh)$$

for any $x, y \in X$ and $g, h \in G$ [15]. We call it the *associated MCQ* of $(X, \{*^g\}_{g\in G})$.

Definition 3.3.2 ([19, 23])**.** Let G be a group with identity element e. A G-family of biquandles is a non-empty set X with two families of binary operations $\underline{*}^g, \overline{*}^g : X \times X \to X$ $(g \in G)$ satisfying the following axioms.

- For any $x \in X$ and $g \in G$,
$$x \,\underline{*}^g\, x = x \,\overline{*}^g\, x.$$
- For any $x, y \in X$ and $g, h \in G$,
$$x \,\underline{*}^{gh}\, y = (x \,\underline{*}^g\, y) \,\underline{*}^h\, (y \,\underline{*}^g\, y), \quad x \,\underline{*}^e\, y = x,$$
$$x \,\overline{*}^{gh}\, y = (x \,\overline{*}^g\, y) \,\overline{*}^h\, (y \,\overline{*}^g\, y), \quad x \,\overline{*}^e\, y = x.$$

- For any $x, y, z \in X$ and $g, h \in G$,

$$(x \mathbin{\underline{*}}^g y) \mathbin{\underline{*}}^h (z \mathbin{\overline{*}}^g y) = (x \mathbin{\underline{*}}^h z) \mathbin{\underline{*}}^{h^{-1}gh} (y \mathbin{\underline{*}}^h z),$$
$$(x \mathbin{\overline{*}}^g y) \mathbin{\underline{*}}^h (z \mathbin{\overline{*}}^g y) = (x \mathbin{\underline{*}}^h z) \mathbin{\overline{*}}^{h^{-1}gh} (y \mathbin{\underline{*}}^h z),$$
$$(x \mathbin{\overline{*}}^g y) \mathbin{\overline{*}}^h (z \mathbin{\overline{*}}^g y) = (x \mathbin{\overline{*}}^h z) \mathbin{\overline{*}}^{h^{-1}gh} (y \mathbin{\underline{*}}^h z).$$

Let R be a ring, G be a group with identity element e and let $f : G \to Z(G)$ be a homomorphism, where $Z(G)$ is the center of G. Let X be a right $R[G]$-module. Then $(X, \{\underline{*}^g\}_{g\in G}, \{\overline{*}^g\}_{g\in G})$ is a G-family of biquandles, called a *G-family of Alexander biquandles*, with $x \mathbin{\underline{*}}^g y = xg + y(f(g) - g)$ and $x \mathbin{\overline{*}}^g y = xf(g)$ [19]. Let $(X, \underline{*}, \overline{*})$ be a biquandle and let m be the type of X. Then $(X, \{\underline{*}^i\}_{i\in\mathbb{Z}_{km}}, \{\overline{*}^i\}_{i\in\mathbb{Z}_{km}})$ is a $\mathbb{Z}_{km}$-family of biquandles for any $k \in \mathbb{Z}_{\geq 0}$ [23]. In particular, when X is an Alexander biquandle, $(X, \{\underline{*}^i\}_{i\in\mathbb{Z}_{km}}, \{\overline{*}^i\}_{i\in\mathbb{Z}_{km}})$ is called a *$\mathbb{Z}_{km}$-family of Alexander biquandles.*

Let $(X, \{\underline{*}^g\}_{g\in G}, \{\overline{*}^g\}_{g\in G})$ be a G-family of biquandles. Then $X \times G = \bigsqcup_{x\in X} \{x\} \times G$ is an MCB with

$$(x, g) \mathbin{\underline{*}} (y, h) := (x \mathbin{\underline{*}}^h y, h^{-1}gh), \qquad (x, g)(x, h) := (x, gh),$$
$$(x, g) \mathbin{\overline{*}} (y, h) := (x \mathbin{\overline{*}}^h y, g)$$

for any $x, y \in X$ and $g, h \in G$ [19, 23]. We call it the *associated MCB* of $(X, \{\underline{*}^g\}_{g\in G}, \{\overline{*}^g\}_{g\in G})$.

3.4 Colorings for handlebody-knots

We introduce a coloring for an S^1-oriented handlebody-link by an MCQ and an MCB. Let $X = \bigsqcup_{\lambda\in\Lambda} G_\lambda$ be an MCQ (resp. MCB) and let D be a diagram of an S^1-oriented handlebody-link H. An *X-coloring* of D is a map $C : \mathcal{A}(D) \to X$ (resp. $\mathcal{SA}(D) \to X$) satisfying the conditions depicted in Figure 3.1 (resp. Figure 3.2) at each crossing and vertex, where $x, y \in X$ and $a, b \in G_\lambda$ for some $\lambda \in \Lambda$. We denote by $\mathrm{Col}_X(D)$ the set of all X-colorings of D.

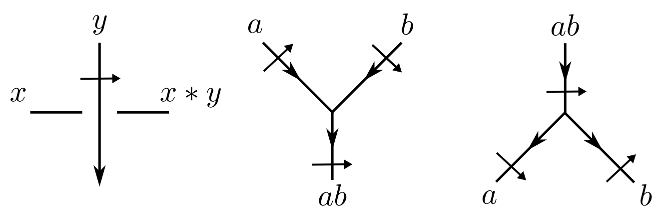


Figure 3.1: An MCQ coloring of D.

Proposition 3.4.1 ([15, 19, 23]). *Let $X = \bigsqcup_{\lambda\in\Lambda} G_\lambda$ be an MCQ or MCB and let D be a diagram of an S^1-oriented handlebody-link H. Let D' be a diagram obtained by applying one of Reidemeister moves to the diagram D once. For an X-coloring C of D, there is a unique X-coloring C' of D' which coincides with C except near the point where the move applied.*

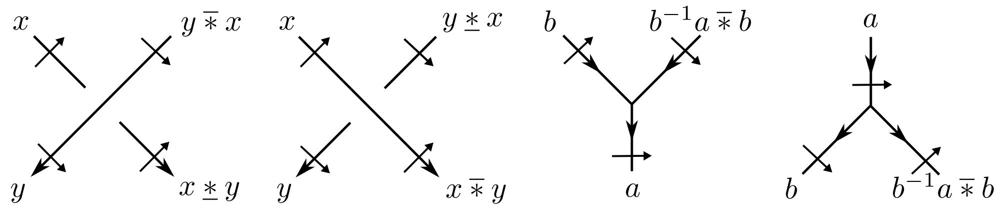


Figure 3.2: An MCB coloring of D.

By this proposition, $\#\mathrm{Col}_X(D)$ is an invariant of H.

Next, we introduce a coloring for an S^1-oriented handlebody-link by a G-family of (bi)quandles. Let D be a diagram of an S^1-oriented handlebody-link H. It is known that the fundamental group $\pi_1(S^3 - H)$ is represented by the arcs, crossings and vertices of D as follows. For a crossing c and a vertex τ of D, we denote by r_c the relation $v_c^{-1}u_cv_c = w_c$ and by r_τ the relation $\alpha_\tau\beta_\tau = \gamma_\tau$, where we denote by $u_c, v_c, w_c, \alpha_\tau, \beta_\tau$ and γ_τ the arcs incident to c or τ as shown in Figure 3.3. The fundamental group $\pi_1(S^3 - H)$ is generated by the arcs x for each $x \in \mathcal{A}(D)$ and has the relations r_c and r_τ for each $c \in C(D)$ and $\tau \in V(D)$, that is, a presentation of $\pi_1(S^3 - H)$ is given by

$$\langle x \ (x \in \mathcal{A}(D)) \mid r_c, r_\tau \ (c \in C(D), \tau \in V(D))\rangle.$$

We call it the *Wirtinger presentation* of $\pi_1(S^3 - H)$ with respect to D.

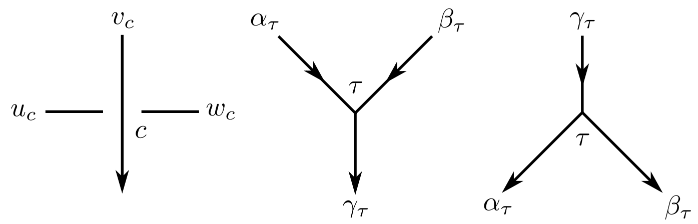


Figure 3.3: Arcs incident to a crossing c or a vertex τ.

Let G be a group and let D be a diagram of an S^1-oriented handlebody-link H. A *G-flow* of D is a map $\phi : \mathcal{A}(D) \to G$ satisfying the conditions depicted in Figure 3.4 at each crossing and vertex. In this thesis, to avoid confusion, we often represent an element of G with an underline. We denote by (D, ϕ), called a *G-flowed diagram* of H, a diagram D given a G-flow ϕ and by $\mathrm{Flow}(D; G)$ the set of all G-flows of D.

We can identify a G-flow ϕ with a group representation of the fundamental group $\pi_1(S^3 - H)$ to G, which is a group homomorphism from $\pi_1(S^3 - H)$ to G. Let D' be a diagram of H obtained by applying one of Reidemeister moves to D once. For any G-flow ϕ of D, there is a unique G-flow ϕ' of D' which coincides with ϕ except near the point where the move applied. Therefore $\#\mathrm{Flow}(D; G)$ is an invariant of H. We call the G-flow ϕ' the *associated G-flow* of ϕ and the G-flowed diagram (D', ϕ') the *associated G-flowed diagram* of (D, ϕ). Since the two G-flows ϕ and ϕ' represent the same group representation ρ, called a G-flow of H, we often use the symbol ρ instead of ϕ and ϕ' and write $\mathrm{Flow}(H;G)$ instead of $\mathrm{Flow}(D;G)$ and $\mathrm{Flow}(D';G)$.

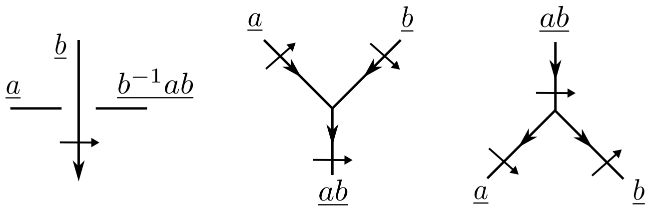


Figure 3.4: A G-flow of D.

Let X be a G-family of quandles (resp. biquandles) and let (D, ρ) be a G-flowed diagram of an S^1-oriented handlebody-link. An *X-coloring* of (D, ρ) is a map $C : \mathcal{A}(D, \rho) \to X$ (resp. $\mathcal{SA}(D, \rho) \to X$) satisfying the conditions depicted in Figure 3.5 (resp. Figure 3.6) at each crossing and vertex. We denote by $\mathrm{Col}_X(D, \rho)$ the set of all X-colorings of (D, ρ). We note that when X is a G-family of Alexander (bi)quandles as a right $R[G]$-module for some ring R, the set $\mathrm{Col}_X(D, \rho)$ is a right R-module with the action $(C \cdot r)(\alpha) := C(\alpha)r$ and the addition $(C + C')(\alpha) := C(\alpha) + C'(\alpha)$ for any $C, C' \in \mathrm{Col}_X(D, \rho)$, $\alpha \in \mathcal{A}(D, \rho)$ (or $\alpha \in \mathcal{SA}(D, \rho)$) and $r \in R$.

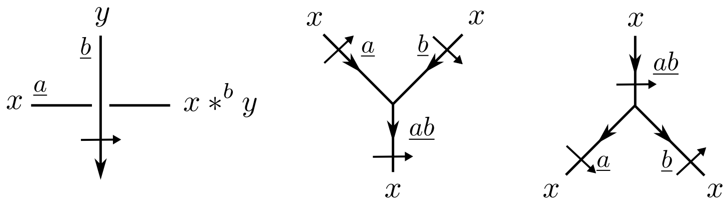


Figure 3.5: A G-family of quandles coloring of (D, ρ).

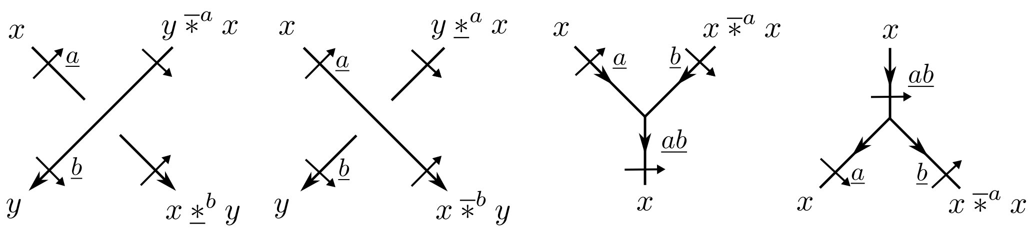


Figure 3.6: A G-family of biquandles coloring of (D, ρ).

Proposition 3.4.2 ([18, 23]). *Let X be a G-family of (bi)quandles, D and D' be diagrams of an S^1-oriented handlebody-link H and let ρ be a G-flow of H. Then there is a one-to-one correspondence between* $\mathrm{Col}_X(D, \rho)$ *and* $\mathrm{Col}_X(D', \rho)$.

By this proposition, for any diagram D of an S^1-oriented handlebody-link H, the multiset $\{\#\mathrm{Col}_X(D, \rho) \mid \rho \in \mathrm{Flow}(H;G)\}$ is an invariant of H.

For example, let (D, ρ) be the $\mathbb{Z}_2$-flowed diagram of the handlebody-knot depicted in Figure 3.7 and let R_3 be the dihedral quandle, that is, $R_3 = \mathbb{Z}_3$ and $x * y = 2y - x$ for any $x, y \in R_3$. We note that $\operatorname{type} R_3 = 2$. Then $(R_3, \{*^i\}_{i \in \mathbb{Z}_2})$ is a $\mathbb{Z}_2$-family of quandles. Therefore the assignment of elements of R_3 to each arc of (D, ρ) as shown in Figure 3.7 is an $(R_3, \{*^i\}_{i \in \mathbb{Z}_2})$-coloring of (D, ρ).

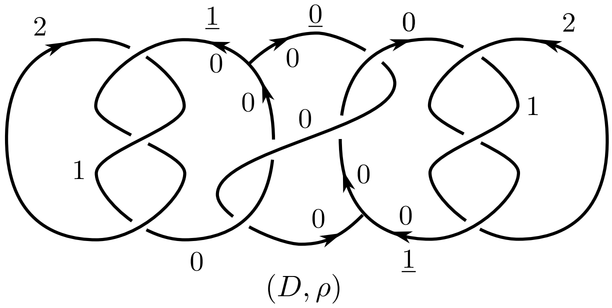


Figure 3.7: A coloring of (D, ρ) by the $\mathbb{Z}_2$-family of quandles $(R_3, \{*^i\}_{i \in \mathbb{Z}_2})$.

For any S^1-oriented handlebody-link, we can regard a G-family of quandles (resp. biquandles) coloring as the associated MCQ (resp. MCB) coloring (see Section 7.3).

Chapter 4

Connected component decompositions of quandles

The work in this chapter is based on a joint work with Yusuke Iijima.

An inner automorphism group of a quandle has an action to the quandle naturally. We call an orbit of the quandle by the action its connected component, which is a subquandle. A quandle is said to be connected if the action is transitive. It is known that all connected quandles of prime square order are Alexander quandles [12].

Any connected component of an Alexander quandle M is isomorphic to $(1-t)M$. Nelson [42] proved that two finite Alexander quandles M and N of the same cardinality are isomorphic if and only if $(1-t)M$ and $(1-t)N$ are isomorphic as modules, and showed connectivity of some Alexander quandles. The numbers of Alexander quandles and connected ones are listed up to order 16 in [42, 43]. In this chapter, for any $f_1(t), \dots, f_k(t) \in \mathbb{Z}[t^{\pm 1}]$, we show that the connected component decomposition of the Alexander quandle $\mathbb{Z}[t^{\pm 1}]/(f_1(t), \dots, f_k(t))$ is $\bigsqcup_{i=0}^{a-1} \mathrm{Orb}(i)$ and that $\mathrm{Orb}(i)$ is isomorphic to $\mathbb{Z}[t^{\pm 1}]/J$ with an explicit ideal J, where $a = \gcd(f_1(1), \dots, f_k(1))$.

A connected subquandle has played an important role in colorings of a knot diagram. However a connected component of a quandle is not a connected quandle in general. In this chapter, we introduce a decomposition of a quandle into the disjoint union of maximal connected subquandles, and show that it is obtained by iterating a connected component decomposition when the quandle is finite, where we note that the decomposition of a finite quandle obtained by iterating a connected component decomposition was introduced in [8, 44]. We also give examples of the decompositions of some quandles. For example, we concretely determine the decompositions of the Alexander quandle $\mathbb{Z}[t^{\pm 1}]/(n_0, t+a)$ for any $n_0 \in \mathbb{Z}_{>0}$ and $a \in \mathbb{Z}$ and the dihedral quandle R_m for any $m \in \mathbb{Z}_{\geq 0}$.

This chapter is organized as follows. In Section 4.1, we recall the definition of a connected component of a quandle. In Section 4.2, we determine the connected component decomposition of the Alexander quandle $\mathbb{Z}[t^{\pm 1}]/(f_1(t), \dots, f_k(t))$. In Section 4.3, we introduce a decomposition of a quandle into the disjoint union of maximal connected subquandles and that it is obtained by iterating a connected component decomposition when the quandle is finite. In Section 4.4, we give examples of the maximal connected subquandle decompositions of some quandles.

4.1 A connected component of a quandle

Let $(X,*)$ be a quandle. A non-empty subset Y of X is called a *subquandle* of X if Y itself is a quandle under $*$. For any non-empty subset Y of X, Y is a subquandle of X if and only if $a*b, a*^{-1}b \in Y$ for any $a,b \in Y$.

For any subset A of X, the minimal subquandle of X including A, denoted by $\langle A\rangle$, is called the subquandle generated by A, that is,

$$\langle A\rangle = \{a *^{k_1} x_1 *^{k_2} \cdots *^{k_n} x_n \in X \mid a, x_1, \ldots, x_n \in A,\ k_1, k_2, \ldots, k_n \in \mathbb{Z}\}.$$

Let X be a quandle. All automorphisms of X form a group under composition of morphisms: $f \cdot g := g \circ f$. This group is called the *automorphism group* of X and denoted $\mathrm{Aut}(X)$. For a subset A of X, we denote by $\mathrm{Inn}(A)$ the subgroup of $\mathrm{Aut}(X)$ generated by $\{S_a \mid a \in A\}$. In particular, $\mathrm{Inn}(X)$ is called the *inner automorphism group* of X. For any $a \in X$ and $g \in \mathrm{Inn}(A)$, we define an action of $\mathrm{Inn}(A)$ on X by $a \cdot g = g(a)$. We say that X is a *connected quandle* if $\mathrm{Inn}(X)$ acts transitively on X. In general, an orbit of X by the action is called a *connected component* of X or an orbit of X simply, and $X = \bigsqcup_{i\in I} X_i$ is called the *connected component decomposition* of X when X_i is a connected component of X for any $i \in I$. In general, a connected component of X is a subquandle of X. We denote by $\mathrm{Orb}_X(a)$ or $\mathrm{Orb}(a)$ the orbit of X containing a.

Example 4.1.1. For any group G, a connected component of $\mathrm{Conj}(G)$ coincides with one of conjugacy classes of G.

In the following, we give a well-known fact with a proof (for example, see [33]).

Lemma 4.1.2. *Let M be an Alexander quandle. Then any connected component of M is isomorphic to $(1-t)M$.*

Proof. Since $0 * x = (1-t)x$ for any $x \in M$, it follows that $(1-t)M \subset \mathrm{Orb}(0)$. On the other hand, for any $z \in \mathrm{Orb}(0)$, there exist $y_1, \ldots, y_n \in M$ such that $z = 0*y_1*y_2*\cdots*y_n = (1-t)y_1*y_2*\cdots*y_n$. Since for any $x, y \in M$, $(1-t)x*y = t(1-t)x + (1-t)y = (1-t)(tx+y)$, we have $z \in (1-t)M$, that is, $(1-t)M \supset \mathrm{Orb}(0)$. Therefore $(1-t)M = \mathrm{Orb}(0)$. Next, we define the map $\phi_a : \mathrm{Orb}(0) \to \mathrm{Orb}(a)$ by $\phi_a(x) = x + a$ for any $a \in M$. For any $(1-t)x \in \mathrm{Orb}(0)$ and $a \in M$, $\phi_a((1-t)x) = (1-t)x + a = ta + (1-t)(x+a) = a*(x+a) \in \mathrm{Orb}(a)$. Hence ϕ_a is well-defined. For any $x, y \in \mathrm{Orb}(0)$ and $a \in M$, $\phi_a(x*y) = (tx + (1-t)y) + a = t(x+a) + (1-t)(y+a) = \phi_a(x) * \phi_a(y)$. Hence ϕ_a is a homomorphism. Similarly, the map $\psi_a : \mathrm{Orb}(a) \to \mathrm{Orb}(0)$ defined by $\psi_a(x) = x - a$ is a homomorphism for any $a \in M$. Since $\phi_a \circ \psi_a = \mathrm{id}_{\mathrm{Orb}(a)}$ and $\psi_a \circ \phi_a = \mathrm{id}_{\mathrm{Orb}(0)}$, ϕ_a is an isomorphism. Therefore $\mathrm{Orb}(0)$ and $\mathrm{Orb}(a)$ are isomorphic for any $a \in M$. $\square$

4.2 The connected component decomposition of an Alexander quandle

In this section, we show that the connected component decomposition of the Alexander quandle $\mathbb{Z}[t^{\pm1}]/(f_1(t), f_2(t), \ldots, f_k(t))$ is $\bigsqcup_{i=0}^{a-1} \mathrm{Orb}(i)$, where

$(f_1(t), f_2(t), \dots, f_k(t))$ is an ideal of $\mathbb{Z}[t^{\pm 1}]$ generated by Laurent polynomials $f_1(t), f_2(t), \dots, f_k(t) \in \mathbb{Z}[t^{\pm 1}]$, and $a = \gcd(f_1(1), f_2(1), \dots, f_k(1))$. Furthermore, we determine the form of all connected components of $\mathbb{Z}[t^{\pm 1}]/(f_1(t), f_2(t), \dots, f_k(t))$.

For any $\alpha(t) \in \mathbb{Z}[t^{\pm 1}]$, we define $C_{\alpha(t)}$ by

$$C_{\alpha(t)} = \{\alpha(t) + a_1 f_1(t) + a_2 f_2(t) + \cdots + a_k f_k(t) \mid a_1, a_2, \dots, a_k \in \mathbb{Z}[t^{\pm 1}]\} \subset \mathbb{Z}[t^{\pm 1}].$$

Then for any $[\alpha(t)] \in \mathbb{Z}[t^{\pm 1}]/(f_1(t), f_2(t), \dots, f_k(t))$, $C_{[\alpha(t)]} := C_{\alpha(t)}$ is well-defined. In this chapter, we often write $\alpha(t)$ for $[\alpha(t)] \in \mathbb{Z}[t^{\pm 1}]/(f_1(t), f_2(t), \dots, f_k(t))$ simply. For any $D \subset \mathbb{Z}[t^{\pm 1}]$, we define $D(1)$ by $D(1) = \{g(1) \mid g(t) \in D\} \subset \mathbb{Z}$. It is easy to see that

$$C_{[\alpha(t)]}(1) = \{\alpha(1) + a_1 f_1(1) + a_2 f_2(1) + \cdots + a_k f_k(1) \mid a_1, a_2, \dots, a_k \in \mathbb{Z}\}$$

for any $[\alpha(t)] \in \mathbb{Z}[t^{\pm 1}]/(f_1(t), f_2(t), \dots, f_k(t))$.

Lemma 4.2.1. *For any elements $[\alpha(t)]$ and $[\beta(t)]$ of the Alexander quandle $\mathbb{Z}[t^{\pm 1}]/(f_1(t), f_2(t), \dots, f_k(t))$, it follows that $C_{[\alpha(t)]*[\beta(t)]}(1) = C_{[\alpha(t)]}(1)$.*

Proof. Since $[\alpha(t)] * [\beta(t)] = [t\alpha(t) + (1-t)\beta(t)]$, we have

$$\begin{aligned}
C_{[\alpha(t)]*[\beta(t)]}(1) &= \{1 \cdot \alpha(1) + (1-1)\beta(1) + a_1 f_1(1) + \cdots + a_k f_k(1) \mid a_1, a_2, \dots, a_k \in \mathbb{Z}\} \\
&= \{\alpha(1) + 0 \cdot \beta(1) + a_1 f_1(1) + \cdots + a_k f_k(1) \mid a_1, a_2, \dots, a_k \in \mathbb{Z}\} \\
&= C_{[\alpha(t)]}(1).
\end{aligned}$$

□

Lemma 4.2.2. *For any elements $[\alpha(t)]$ and $[\beta(t)]$ of the Alexander quandle $\mathbb{Z}[t^{\pm 1}]/(f_1(t), f_2(t), \dots, f_k(t))$, it follows that $C_{[\alpha(t)]}(1) = C_{[\beta(t)]}(1)$ if and only if $\mathrm{Orb}([\alpha(t)]) = \mathrm{Orb}([\beta(t)])$.*

Proof. Suppose that $C_{[\alpha(t)]}(1) = C_{[\beta(t)]}(1)$. Then we have

$$\begin{aligned}
&\{\alpha(1) + l_1 f_1(1) + l_2 f_2(1) + \cdots + l_k f_k(1) \mid l_1, l_2, \dots, l_k \in \mathbb{Z}\} \\
&= C_{[\alpha(t)]}(1) \\
&= C_{[\beta(t)]}(1) \\
&= \{\beta(1) + l_1 f_1(1) + l_2 f_2(1) + \cdots + l_k f_k(1) \mid l_1, l_2, \dots, l_k \in \mathbb{Z}\}.
\end{aligned}$$

Hence there exist $l_1, l_2, \dots, l_k \in \mathbb{Z}$ such that $\alpha(1) = \beta(1) + l_1 f_1(1) + l_2 f_2(1) + \cdots + l_k f_k(1)$. We put $\widetilde{\alpha}(t)$ and $\widetilde{\beta}(t)$ by $\alpha(t) = (1-t)\widetilde{\alpha}(t) + \alpha(1)$ and $\beta(t) = (1-t)\widetilde{\beta}(t) + \beta(1)$ respectively. We also put $\widetilde{f}_i(t)$ by $f_i(t) = (1-t)\widetilde{f}_i(t) + f_i(1)$ for any $i = 1, 2, \dots, k$ and $\gamma(t) := \widetilde{\beta}(t) + \beta(1) - t\widetilde{\alpha}(t) + \sum_{i=1}^{k} l_i t \widetilde{f}_i(t)$. Since $[\sum_{i=1}^{k} l_i t f_i(t)] = [0]$

in $\mathbb{Z}[t^{\pm 1}]/(f_1(t), f_2(t), \ldots, f_k(t))$, we have

$$
\begin{aligned}
&[(1-t)\gamma(t)] \\
&= [(1-t)(\widetilde{\beta}(t) + \beta(1) - t\widetilde{\alpha}(t) + \sum_{i=1}^{k} l_i t \widetilde{f}_i(t))] \\
&= [(1-t)\widetilde{\beta}(t) + (1-t)\beta(1) - t(1-t)\widetilde{\alpha}(t) + \sum_{i=1}^{k} l_i t(1-t)\widetilde{f}_i(t)] \\
&= [(1-t)\widetilde{\beta}(t) + (1-t)\beta(1) - t(1-t)\widetilde{\alpha}(t) + \sum_{i=1}^{k} l_i t(1-t)\widetilde{f}_i(t) - \sum_{i=1}^{k} l_i t f_i(t)] \\
&= [(1-t)\widetilde{\beta}(t) + \beta(1) - t\beta(1) - t(1-t)\widetilde{\alpha}(t) + \sum_{i=1}^{k} l_i t((1-t)\widetilde{f}_i(t) - f_i(t))] \\
&= [\beta(t) - t(1-t)\widetilde{\alpha}(t) - t\beta(1) + \sum_{i=1}^{k} l_i t(-f_i(1))] \\
&= [\beta(t) - t(1-t)\widetilde{\alpha}(t) - t\alpha(1)] \\
&= [\beta(t) - t\alpha(t)].
\end{aligned}
$$

Hence $[\alpha(t)] * [\gamma(t)] = [t\alpha(t) + (1-t)\gamma(t)] = [\beta(t)]$, which implies $\mathrm{Orb}([\alpha(t)]) = \mathrm{Orb}([\beta(t)])$. Next, suppose that $\mathrm{Orb}([\alpha(t)]) = \mathrm{Orb}([\beta(t)])$. There exist $[\gamma_1(t)], [\gamma_2(t)], \ldots, [\gamma_l(t)] \in \mathbb{Z}[t^{\pm 1}]/(f_1(t), f_2(t), \ldots, f_k(t))$ and $\epsilon_1, \epsilon_2, \ldots, \epsilon_l \in \mathbb{Z}$ such that $[\alpha(t)] *^{\epsilon_1} [\gamma_1(t)] *^{\epsilon_2} \cdots *^{\epsilon_l} [\gamma_l(t)] = [\beta(t)]$. By Lemma 4.2.1, we have $C_{[\alpha(t)]}(1) = C_{[\beta(t)]}(1)$. $\square$

Then we have the following lemma.

Lemma 4.2.3. *For any elements $[\alpha(t)]$ and $[\beta(t)]$ of the Alexander quandle $\mathbb{Z}[t^{\pm 1}]/(f_1(t), f_2(t), \ldots, f_k(t))$, it follows that $\mathrm{Orb}([\alpha(t)]) = \mathrm{Orb}([\beta(t)])$ if and only if $\alpha(1) \equiv \beta(1) \mod a$, where $a = \gcd(f_1(1), f_2(1), \ldots, f_k(1))$.*

Proof. Suppose that $\mathrm{Orb}([\alpha(t)]) = \mathrm{Orb}([\beta(t)])$. By Lemma 4.2.2, $C_{[\alpha(t)]}(1) = C_{[\beta(t)]}(1)$. Since $f_1(1)\mathbb{Z} + f_2(1)\mathbb{Z} + \cdots + f_k(1)\mathbb{Z} = a\mathbb{Z}$, we have

$$
\begin{aligned}
\{\alpha(1) + la \mid l \in \mathbb{Z}\} &= \{\alpha(1) + l_1 f_1(1) + l_2 f_2(1) + \cdots + l_k f_k(1) \mid l_1, l_2, \ldots, l_k \in \mathbb{Z}\} \\
&= C_{[\alpha(t)]}(1) \\
&= C_{[\beta(t)]}(1) \\
&= \{\beta(1) + l_1 f_1(1) + l_2 f_2(1) + \cdots + l_k f_k(1) \mid l_1, l_2, \ldots, l_k \in \mathbb{Z}\} \\
&= \{\beta(1) + la \mid l \in \mathbb{Z}\}.
\end{aligned}
$$

Hence we obtain $\alpha(1) \equiv \beta(1) \mod a$. On the other hand, suppose that $\alpha(1) \equiv \beta(1)$

mod a. Since $f_1(1)\mathbb{Z} + f_2(1)\mathbb{Z} + \cdots + f_k(1)\mathbb{Z} = a\mathbb{Z}$, we have

$$\begin{aligned} C_{[\alpha(t)]}(1) &= \{\alpha(1) + l_1 f_1(1) + l_2 f_2(1) + \cdots + l_k f_k(1) \,|\, l_1, l_2, \ldots, l_k \in \mathbb{Z}\} \\ &= \{\alpha(1) + la \,|\, l \in \mathbb{Z}\} \\ &= \{\beta(1) + la \,|\, l \in \mathbb{Z}\} \\ &= \{\beta(1) + l_1 f_1(1) + l_2 f_2(1) + \cdots + l_k f_k(1) \,|\, l_1, l_2, \ldots, l_k \in \mathbb{Z}\} \\ &= C_{[\beta(t)]}(1). \end{aligned}$$

By Lemma 4.2.2, we obtain $\mathrm{Orb}([\alpha(t)]) = \mathrm{Orb}([\beta(t)])$. □

Then we have the following theorem.

Theorem 4.2.4. *Let M be the Alexander quandle $\mathbb{Z}[t^{\pm 1}]/(f_1(t), f_2(t), \ldots, f_k(t))$ and let $a = \gcd(f_1(1), f_2(1), \ldots, f_k(1))$. Then the following hold.*

1. *The connected component decomposition of M is given by*

$$M = \bigsqcup_{i=0}^{a-1} \mathrm{Orb}(i),$$

where

$$\begin{aligned} \mathrm{Orb}(i) &= \{[g(t)] \mid g(t) \in \mathbb{Z}[t^{\pm 1}],\ g(1) \equiv i \mod a\} \\ &= \{[i + (1-t)g(t) + aj] \mid g(t) \in \mathbb{Z}[t^{\pm 1}],\ j \in \mathbb{Z}\}. \end{aligned}$$

2. *For any $j = 0, 1, \ldots, a-1$, it follows that*

$$\mathrm{Orb}(j) \cong \mathbb{Z}[t^{\pm 1}]/((f_1(t), f_2(t), \ldots, f_k(t)) + I),$$

where we define $\widetilde{f}_i(t)$ by $f_i(t) = (1-t)\widetilde{f}_i(t) + f_i(1)$ for any $i = 1, 2, \ldots, k$, and $I = \{\sum_{i=1}^{k} a_i \widetilde{f}_i(t) \mid a_i \in \mathbb{Z}[t^{\pm 1}],\ \sum_{i=1}^{k} a_i f_i(1) = 0\}$.

Proof. 1. By Lemma 4.2.3, $M = \bigsqcup_{i=0}^{a-1} \mathrm{Orb}(i)$ is the connected component decomposition of M, and we have $\mathrm{Orb}(i) = \{[g(t)] \mid g(t) \in \mathbb{Z}[t^{\pm 1}],\ g(1) \equiv i \mod a\}$ immediately. There exist $\widetilde{g}(t) \in \mathbb{Z}[t^{\pm 1}]$ and $j \in \mathbb{Z}$ such that $g(t) = g(1) + (1-t)\widetilde{g}(t) = i + (1-t)\widetilde{g}(t) + aj$ if and only if $g(1) \equiv i \mod a$. Hence we have $\mathrm{Orb}(i) = \{[i + (1-t)g(t) + aj] \mid g(t) \in \mathbb{Z}[t^{\pm 1}],\ j \in \mathbb{Z}\}$.

2. Let $J = (f_1(t), f_2(t), \ldots, f_k(t))$. We define the $\mathbb{Z}[t^{\pm 1}]$-homomorphism $\phi : \mathbb{Z}[t^{\pm 1}] \to (1-t)M$ by $\phi(x) = (1-t)x + J$. It is clear that $J \subset \ker(\phi)$.

For any $\sum_{i=1}^{k} a_i\widetilde{f_i}(t) \in I$, we have

$$\begin{aligned}\phi(\sum_{i=1}^{k} a_i\widetilde{f_i}(t)) &= (1-t)\sum_{i=1}^{k} a_i\widetilde{f_i}(t) + J\\ &= \sum_{i=1}^{k} a_i(1-t)\widetilde{f_i}(t) + \sum_{i=1}^{k} a_i f_i(1) + J\\ &= \sum_{i=1}^{k} a_i((1-t)\widetilde{f_i}(t) + f_i(1)) + J\\ &= \sum_{i=1}^{k} a_i f_i(t) + J,\end{aligned}$$

which implies that $I \subset \ker(\phi)$. Hence we obtain $J + I \subset \ker(\phi)$. On the other hand, let $g(t) \in \ker(\phi)$. Since $\phi(g(t)) = (1-t)g(t) + J = J$, there exist $h_1(t), h_2(t), \ldots, h_k(t) \in \mathbb{Z}[t^{\pm 1}]$ such that $(1-t)g(t) = \sum_{i=1}^{k} h_i(t)f_i(t)$. When we put $t = 1$, we have $0 = \sum_{i=1}^{k} h_i(1)f_i(1)$. For any $i = 1, 2, \ldots, k$, we define $\widetilde{h_i}(t)$ by $h_i(t) = (1-t)\widetilde{h_i}(t) + h_i(1)$. Since $\sum_{i=1}^{k} h_i(1)f_i(1) = 0$, we have

$$\begin{aligned}(1-t)g(t) &= \sum_{i=1}^{k} h_i(t)f_i(t)\\ &= \sum_{i=1}^{k}((1-t)\widetilde{h_i}(t)f_i(t) + (1-t)h_i(1)\widetilde{f_i}(t) + h_i(1)f_i(1))\\ &= (1-t)\sum_{i=1}^{k}\widetilde{h_i}(t)f_i(t) + (1-t)\sum_{i=1}^{k} h_i(1)\widetilde{f_i}(t) + \sum_{i=1}^{k} h_i(1)f_i(1)\\ &= (1-t)\sum_{i=1}^{k}\widetilde{h_i}(t)f_i(t) + (1-t)\sum_{i=1}^{k} h_i(1)\widetilde{f_i}(t).\end{aligned}$$

Hence we have $g(t) = \sum_{i=1}^{k}\widetilde{h_i}(t)f_i(t) + \sum_{i=1}^{k} h_i(1)\widetilde{f_i}(t)$. Since $\sum_{i=1}^{k} h_i(1)f_i(1) = 0$, we have $\sum_{i=1}^{k} h_i(1)\widetilde{f_i}(t) \in I$, that is, $g(t) \in J + I$. Hence we obtain $J + I \supset \ker(\phi)$. Obviously, ϕ is a surjection. By the homomorphism theorem, $\widetilde{\phi} : \mathbb{Z}[t^{\pm 1}]/(J + I) \to (1-t)M$ is a $\mathbb{Z}[t^{\pm 1}]$-isomorphism, which is an isomorphism as quandles. By Lemma 4.1.2, it follows that $\mathrm{Orb}(j) \cong \mathbb{Z}[t^{\pm 1}]/((f_1(t), f_2(t), \ldots, f_k(t)) + I)$ for any $j = 0, 1, \ldots, a-1$.

□

By Theorem 4.2.4, we obtain the following corollaries, where we note that the dihedral quandle R_m is isomorphic to the Alexander quandle $\mathbb{Z}[t^{\pm 1}]/(m, t+1)$ for any $m \in \mathbb{Z}_{\geq 0}$.

Corollary 4.2.5. *For any $m \in \mathbb{Z}_{>0}$, R_m is a connected dihedral quandle if and only if m is an odd number. Furthermore, when m is an even number, $R_m = \mathrm{Orb}(0) \sqcup \mathrm{Orb}(1) = \{0, 2, \ldots, m-2\} \sqcup \{1, 3, \ldots, m-1\}$ is the connected component decomposition of R_m.*

Corollary 4.2.6. *Let $f_1(t), f_2(t), \ldots, f_k(t) \in \mathbb{Z}[t^{\pm 1}]$. Then the Alexander quandle $\mathbb{Z}[t^{\pm 1}]/(f_1(t), f_2(t), \ldots, f_k(t))$ is connected if and only if* $\gcd(f_1(1), f_2(1), \ldots, f_k(1)) = 1$.

For example, the tetrahedral quandle $\mathbb{Z}[t^{\pm 1}]/(2, t^2+t+1)$ is connected by Corollary 4.2.6.

Corollary 4.2.7. *Let $a, m \in \mathbb{Z}$ and let $n = m/\gcd(m, 1+a)$. Then any connected component of the Alexander quandle $\mathbb{Z}[t^{\pm 1}]/(m, t+a)$ is isomorphic to $\mathbb{Z}[t^{\pm 1}]/(n, t+a)$.*

Proof. By Theorem 4.2.4, any connected component of $\mathbb{Z}[t^{\pm 1}]/(m, t+a)$ is isomorphic to $\mathbb{Z}[t^{\pm 1}]/((m, t+a)+I)$, where $I = \{-a_2 \mid a_1, a_2 \in \mathbb{Z}[t^{\pm 1}],\ a_1 m + a_2(1+a) = 0\}$. For any $-x_2 \in I$, there exists $x_1 \in \mathbb{Z}[t^{\pm 1}]$ such that $x_1 m + x_2(1+a) = 0$. Then we have $x_1 n + x_2(1+a)/\gcd(m, 1+a) = 0$. Since n and $(1+a)/\gcd(m, 1+a)$ are relatively prime, x_2 is devisible by n. Hence we obtain $I \subset n\mathbb{Z}[t^{\pm 1}]$. On the other hand, for any $x_2 = ns \in n\mathbb{Z}[t^{\pm 1}]$, there exists $x_1 = -s(1+a)/\gcd(m, 1+a) \in \mathbb{Z}[t^{\pm 1}]$ such that $x_1 m + x_2(1+a) = 0$. Hence $-x_2 \in I$, that is, $I \supset n\mathbb{Z}[t^{\pm 1}]$. Therefore $I = n\mathbb{Z}[t^{\pm 1}]$. Since m is divisible by n, we have $\mathbb{Z}[t^{\pm 1}]/((m, t+a)+I) = \mathbb{Z}[t^{\pm 1}]/(m, n, t+a) = \mathbb{Z}[t^{\pm 1}]/(n, t+a)$. □

Corollary 4.2.8. *If m is an even number, then any connected component of R_m is isomorphic to $R_{m/2}$.*

4.3 The maximal connected subquandle decomposition

In this section, we consider a decomposition of a quandle into the disjoint union of maximal connected subquandles, and review that it is uniquely obtained by iterating a connected component decomposition when the quandle is finite. We remark that $\{(1,2,3), (1,3,2)\}$ is a connected component of $\mathrm{Conj}(S_3)$, but not a connected subquandle of it, where S_3 is a symmetric group of degree 3.

Let X be a quandle and let A be a connected subquandle of X. We say that A is a *maximal connected subquandle* of X when any connected subquandle of X including A is only A. We say that $X = \bigsqcup_{i \in I} A_i$ is the *maximal connected subquandle decomposition* of X when each A_i is a maximal connected subquandle of X.

Theorem 4.3.1 ([13])**.** *Any quandle has a unique maximal connected subquandle decomposition.*

For a quandle X, it is easy to see that any connected subquandle of X is included in some connected component of X. Therefore if a connected component of X is a connected subquandle of X, then it is a maximal connected subquandle of X.

Let X be a quandle and let $\mathcal{P}_{\mathsf{Qnd}}(X)$ be the set of all subquandles of X. For any $\mathcal{A} \subset \mathcal{P}_{\mathsf{Qnd}}(X)$, we define $D(\mathcal{A}) := \bigcup_{A \in \mathcal{A}} \{\mathrm{Orb}_A(a) \mid a \in A\}$. It is easy to see that $\bigcup_{A \in \mathcal{A}} A = \bigcup_{A \in D(\mathcal{A})} A$. We put $D^0(\mathcal{A}) := \mathcal{A}$ and $D^{k+1}(\mathcal{A}) := D(D^k(\mathcal{A}))$ for any $k \in \mathbb{Z}_{\geq 0}$.

Theorem 4.3.2 ([13])**.** *Let X be a quandle. If there exists $n \in \mathbb{Z}_{\geq 0}$ such that $D^n(\{X\}) = D^{n+1}(\{X\})$, then $X = \bigsqcup_{A \in D^n(\{X\})} A$ is the maximal connected subquandle decomposition of X. In particular, if X is a finite quandle, then there exists*

$n \in \mathbb{Z}_{\geq 0}$ such that $X = \bigsqcup_{A \in D^n(\{X\})} A$ is the maximal connected subquandle decomposition of X.

By Lemma 4.1.2 and Theorem 4.3.2, the following corollary holds immediately.

Corollary 4.3.3. *All maximal connected subquandles of a finite Alexander quandle are isomorphic.*

For a quandle X, we denote by $\mathrm{depth}(X)$ the minimal number of n satisfying $X = \bigsqcup_{A \in D^n(\{X\})} A$ is the maximal connected subquandle decomposition of X. It is called the *subquandle depth* of X in [44]. Obviously, X is a connected quandle if and only if $\mathrm{depth}(X) = 0$.

4.4 Examples of the maximal connected subquandle decomposition

In this section, we give examples of the maximal connected subquandle decompositions of some quandles.

Let S_n be a symmetric group of degree n. We consider connectivity of $\mathrm{Conj}(S_n)$. By Example 4.1.1, a connected component of $\mathrm{Conj}(S_n)$ coincides with one of conjugacy classes of S_n. We denote by $C(a)$ the conjugacy class of S_n containing a. We note that two elements of S_n are conjugate if and only if their cyclic types coincide.

Example 4.4.1. (1) We show that the maximal connected subquandle decomposition of S_3 is

$$S_3 = \{(1\,2\,3)\} \sqcup \{(1\,3\,2)\} \sqcup C((1\,2)) \sqcup C(e).$$

$S_3 = C((1\,2\,3)) \sqcup C((1\,2)) \sqcup C(e)$ is the connected component decomposition of S_3. Furthermore, $C((1\,2))$ and $C(e)$ are connected quandles, and $C((1\,2\,3)) = \{(1\,2\,3)\} \sqcup \{(1\,3\,2)\}$ is the connected component decomposition of $C((1\,2\,3))$. Therefore

$$S_3 = \{(1\,2\,3)\} \sqcup \{(1\,3\,2)\} \sqcup C((1\,2)) \sqcup C(e)$$

is the maximal connected subquandle decomposition of S_3, and we have $\mathrm{depth}(S_3) = 2$.

(2) We show that the maximal connected subquandle decomposition of S_4 is

$$\begin{aligned} S_4 = {} & C((1\,2\,3\,4)) \\ & \sqcup \{(1\,2\,3), (1\,4\,2), (1\,3\,4), (2\,4\,3)\} \sqcup \{(1\,3\,2), (1\,2\,4), (1\,4\,3), (2\,3\,4)\} \\ & \sqcup \{(1\,2)(3\,4)\} \sqcup \{(1\,3)(2\,4)\} \sqcup \{(1\,4)(2\,3)\} \sqcup C((1\,2)) \sqcup C(e). \end{aligned}$$

$S_4 = C((1\,2\,3\,4)) \sqcup C((1\,2\,3)) \sqcup C((1\,2)(3\,4)) \sqcup C((1\,2)) \sqcup C(e)$ is the connected component decomposition of S_4. Furthermore, $C((1\,2\,3\,4))$, $C((1\,2))$ and $C(e)$ are connected quandles, and $C((1\,2\,3)) = \{(1\,2\,3), (1\,4\,2), (1\,3\,4), (2\,4\,3)\} \sqcup \{(1\,3\,2), (1\,2\,4), (1\,4\,3), (2\,3\,4)\}$ and $C((1\,2)(3\,4)) = \{(1\,2)(3\,4)\} \sqcup \{(1\,3)(2\,4)\} \sqcup$

$\{(1\,4)(2\,3)\}$ are the connected component decompositions of $C((1\,2\,3))$ and $C((1\,2)(3\,4))$ respectively. Since any connected component of $C((1\,2\,3))$ and $C((1\,2)(3\,4))$ is connected,

$$\begin{aligned} S_4 = {} & C((1\,2\,3\,4)) \\ & \sqcup\{(1\,2\,3),(1\,4\,2),(1\,3\,4),(2\,4\,3)\}\sqcup\{(1\,3\,2),(1\,2\,4),(1\,4\,3),(2\,3\,4)\} \\ & \sqcup\{(1\,2)(3\,4)\}\sqcup\{(1\,3)(2\,4)\}\sqcup\{(1\,4)(2\,3)\}\sqcup C((1\,2))\sqcup C(e) \end{aligned}$$

is the maximal connected subquandle decomposition of S_4, and we have $\mathrm{depth}(S_4)=2$.

(3) We show that the maximal connected subquandle decomposition of S_5 is

$$\begin{aligned} S_5 = {} & \left\{ \begin{array}{l} (1\,2\,3\,4\,5),(1\,2\,5\,3\,4),(1\,2\,4\,5\,3),(1\,3\,2\,5\,4),(1\,3\,4\,2\,5),(1\,3\,5\,4\,2), \\ (1\,4\,3\,5\,2),(1\,4\,5\,2\,3),(1\,4\,2\,3\,5),(1\,5\,3\,2\,4),(1\,5\,2\,4\,3),(1\,5\,4\,3\,2) \end{array} \right\} \\ & \sqcup \left\{ \begin{array}{l} (1\,2\,3\,5\,4),(1\,2\,5\,4\,3),(1\,2\,4\,3\,5),(1\,3\,2\,4\,5),(1\,3\,4\,5\,2),(1\,3\,5\,2\,4), \\ (1\,4\,3\,2\,5),(1\,4\,5\,3\,2),(1\,4\,2\,5\,3),(1\,5\,3\,4\,2),(1\,5\,2\,3\,4),(1\,5\,4\,2\,3) \end{array} \right\} \\ & \sqcup C((1\,2\,3\,4))\sqcup C((1\,2\,3))\sqcup C((1\,2)(3\,4\,5))\sqcup C((1\,2)(3\,4))\sqcup C((1\,2))\sqcup C(e). \end{aligned}$$

$S_5 = C((1\,2\,3\,4\,5))\sqcup C((1\,2\,3\,4))\sqcup C((1\,2\,3))\sqcup C((1\,2)(3\,4\,5))\sqcup C((1\,2)(3\,4))\sqcup C((1\,2))\sqcup C(e)$ is the connected component decomposition of S_5. Furthermore, $C((1\,2\,3\,4))$, $C((1\,2\,3))$, $C((1\,2)(3\,4\,5))$, $C((1\,2)(3\,4))$, $C((1\,2))$ and $C(e)$ are connected quandles, and

$$\begin{aligned} & C((1\,2\,3\,4\,5)) \\ & = \left\{ \begin{array}{l} (1\,2\,3\,4\,5),(1\,2\,5\,3\,4),(1\,2\,4\,5\,3),(1\,3\,2\,5\,4),(1\,3\,4\,2\,5),(1\,3\,5\,4\,2), \\ (1\,4\,3\,5\,2),(1\,4\,5\,2\,3),(1\,4\,2\,3\,5),(1\,5\,3\,2\,4),(1\,5\,2\,4\,3),(1\,5\,4\,3\,2) \end{array} \right\} \\ & \quad \sqcup \left\{ \begin{array}{l} (1\,2\,3\,5\,4),(1\,2\,5\,4\,3),(1\,2\,4\,3\,5),(1\,3\,2\,4\,5),(1\,3\,4\,5\,2),(1\,3\,5\,2\,4), \\ (1\,4\,3\,2\,5),(1\,4\,5\,3\,2),(1\,4\,2\,5\,3),(1\,5\,3\,4\,2),(1\,5\,2\,3\,4),(1\,5\,4\,2\,3) \end{array} \right\} \end{aligned}$$

is the connected component decomposition of $C((1\,2\,3\,4\,5))$. Since any connected component of $C((1\,2\,3\,4\,5))$ is connected,

$$\begin{aligned} S_5 = {} & \left\{ \begin{array}{l} (1\,2\,3\,4\,5),(1\,2\,5\,3\,4),(1\,2\,4\,5\,3),(1\,3\,2\,5\,4),(1\,3\,4\,2\,5),(1\,3\,5\,4\,2), \\ (1\,4\,3\,5\,2),(1\,4\,5\,2\,3),(1\,4\,2\,3\,5),(1\,5\,3\,2\,4),(1\,5\,2\,4\,3),(1\,5\,4\,3\,2) \end{array} \right\} \\ & \sqcup \left\{ \begin{array}{l} (1\,2\,3\,5\,4),(1\,2\,5\,4\,3),(1\,2\,4\,3\,5),(1\,3\,2\,4\,5),(1\,3\,4\,5\,2),(1\,3\,5\,2\,4), \\ (1\,4\,3\,2\,5),(1\,4\,5\,3\,2),(1\,4\,2\,5\,3),(1\,5\,3\,4\,2),(1\,5\,2\,3\,4),(1\,5\,4\,2\,3) \end{array} \right\} \\ & \sqcup C((1\,2\,3\,4))\sqcup C((1\,2\,3))\sqcup C((1\,2)(3\,4\,5))\sqcup C((1\,2)(3\,4))\sqcup C((1\,2))\sqcup C(e) \end{aligned}$$

is the maximal connected subquandle decomposition of S_5, and we have $\mathrm{depth}(S_5)=2$.

Example 4.4.2. We show that the maximal connected subquandle decomposition of the dihedral quandle R_0 is

$$R_0 = \bigsqcup_{i\in\mathbb{Z}}\{i\}.$$

We note that R_0 is isomorphic to the Alexander quandle $\mathbb{Z}[t^{\pm 1}]/(t+1)$. By Theorem 4.2.4, the connected component decomposition of R_0 is

$$R_0 = \mathrm{Orb}(0) \sqcup \mathrm{Orb}(1) = \{i \mid i : \text{even}\} \sqcup \{i \mid i : \text{odd}\}.$$

Since each connected component is isomorphic to R_0 by Corollary 4.2.8, we have $D^n(\{R_0\}) = \{\{2^n j + i \mid j \in \mathbb{Z}\} \mid i = 0, 1, \ldots, 2^n - 1\}$ for any $n \in \mathbb{Z}_{\geq 0}$ by iterating a connected component decomposition. Hence for any $a, b \in R_0$, there exists $l \in \mathbb{Z}_{>0}$ such that a and b are in distinct elements of $D^l(\{R_0\})$. Since any connected subquandle is included in a connected component, any connected subquandle of R_0 is included in an element of $D^l(\{R_0\})$. Therefore a and b are in distinct maximal connected subquandles of R_0, which implies that $R_0 = \bigsqcup_{i \in \mathbb{Z}} \{i\}$ is the maximal connected subquandle decomposition of R_0, and $\mathrm{depth}(R_0) = \infty$.

Example 4.4.3. We consider the maximal connected subquandle decomposition of the Alexander quandle $\mathbb{Z}[t^{\pm 1}]/(6, t^2+t+1)$. Since $\gcd(6, 1^2+1+1) = 3$, $\mathbb{Z}[t^{\pm 1}]/(6, t^2 + t + 1) = \mathrm{Orb}(0) \sqcup \mathrm{Orb}(1) \sqcup \mathrm{Orb}(2)$ is the connected component decomposition of $\mathbb{Z}[t^{\pm 1}]/(6, t^2 + t + 1)$, and

$$\begin{aligned}
\mathrm{Orb}(0) &= \{[g(t)] \mid g(t) \in \mathbb{Z}[t^{\pm 1}],\ g(1) \equiv 0 \mod 3\} \\
&= \left\{ \begin{array}{l} 0, 3, 3t, 1+2t, 1+5t, 2+t, 2+4t, \\ 3+3t, 4+2t, 4+5t, 5+t, 5+4t \end{array} \right\}, \\
\mathrm{Orb}(1) &= \{[g(t)] \mid g(t) \in \mathbb{Z}[t^{\pm 1}],\ g(1) \equiv 1 \mod 3\} \\
&= \left\{ \begin{array}{l} 1, 4, t, 4t, 1+3t, 2+2t, 2+5t, \\ 3+t, 3+4t, 4+3t, 5+2t, 5+5t \end{array} \right\}, \\
\mathrm{Orb}(2) &= \{[g(t)] \mid g(t) \in \mathbb{Z}[t^{\pm 1}],\ g(1) \equiv 2 \mod 3\} \\
&= \left\{ \begin{array}{l} 2, 5, 2t, 5t, 1+t, 1+4t, 2+3t, \\ 3+2t, 3+5t, 4+t, 4+4t, 5+3t \end{array} \right\}
\end{aligned}$$

by Theorem 4.2.4.

Next, by Theorem 4.2.4, for any $i = 0, 1, 2$, $\mathrm{Orb}(i)$ is isomorphic to $\mathbb{Z}[t^{\pm 1}]/((6, t^2 + t + 1) + I)$, where

$$\begin{aligned}
I &= \{-a_2(t+2) \mid a_1, a_2 \in \mathbb{Z}[t^{\pm 1}],\ 6a_1 + 3a_2 = 0\} \\
&= \{-a_2(t+2) \mid a_1, a_2 \in \mathbb{Z}[t^{\pm 1}],\ a_2 = -2a_1\} \\
&= 2(t+2)\mathbb{Z}[t^{\pm 1}],
\end{aligned}$$

which implies that $\mathbb{Z}[t^{\pm 1}]/((6, t^2 + t + 1) + I) = \mathbb{Z}[t^{\pm 1}]/(6, 2(t+2), t^2 + t + 1)$. By Theorem 4.2.4, we obtain that

$$\mathbb{Z}[t^{\pm 1}]/(6, 2(t+2), t^2 + t + 1) = \{0, 3, 2+t, 5+t\} \sqcup \{1, 4, t, 3+t\} \sqcup \{2, 5, 1+t, 4+t\}$$

is the connected component decomposition of $\mathbb{Z}[t^{\pm 1}]/(6, 2(t+2), t^2 + t + 1)$. By the proof of Theorem 4.2.4, the map $\widetilde{\phi} : \mathbb{Z}[t^{\pm 1}]/(6, 2(t+2), t^2 + t + 1) \to (1 - t)(\mathbb{Z}[t^{\pm 1}]/(6, t^2 + t + 1))$ defined by $\widetilde{\phi}(x) = (1-t)x$ is an isomorphism. Hence, by the proof of Lemma 4.1.2,

$$\begin{aligned}
\mathrm{Orb}(0) &= (1-t)(\mathbb{Z}[t^{\pm 1}]/(6, t^2 + t + 1)) \\
&= \{0, 3, 3t, 3+3t\} \sqcup \{1+5t, 1+2t, 4+2t, 4+5t\} \sqcup \{2+4t, 2+t, 5+t, 5+4t\}
\end{aligned}$$

is the connected component decomposition of Orb(0). Furthermore, for any $i = 1, 2$, the map $\phi_i : \mathrm{Orb}(0) \to \mathrm{Orb}(i)$ defined by $\phi_i(x) = x + i$ is an isomorphism by the proof of Lemma 4.1.2. Therefore

$$\mathrm{Orb}(1) = \{1, 4, 1 + 3t, 4 + 3t\} \sqcup \{2 + 5t, 2 + 2t, 5 + 2t, 5 + 5t\} \sqcup \{3 + 4t, 3 + t, t, 4t\}$$

and

$$\mathrm{Orb}(2) = \{2, 5, 2 + 3t, 5 + 3t\} \sqcup \{3 + 5t, 3 + 2t, 2t, 5t\} \sqcup \{4 + 4t, 4 + t, 1 + t, 1 + 4t\}$$

are the connected component decompositions of Orb(1) and Orb(2) respectively.

Finally, by Theorem 4.2.4, any connected component of $\mathbb{Z}[t^{\pm 1}]/(6, 2(t+2), t^2+t+1)$ is isomorphic to $\mathbb{Z}[t^{\pm 1}]/((6, 2(t + 2), t^2 + t + 1) + I')$, where

$$\begin{aligned} I' &= \{-2a_2 - a_3(t + 2) \mid a_1, a_2, a_3 \in \mathbb{Z}[t^{\pm 1}],\ 6a_1 + 6a_2 + 3a_3 = 0\} \\ &= \{-2a_2 - a_3(t + 2) \mid a_1, a_2, a_3 \in \mathbb{Z}[t^{\pm 1}],\ a_3 = -2(a_1 + a_2)\} \\ &= \{-2a_2 + 2(a_1 + a_2)(t + 2) \mid a_1, a_2 \in \mathbb{Z}[t^{\pm 1}]\} \\ &= 2\mathbb{Z}[t^{\pm 1}], \end{aligned}$$

which implies that $\mathbb{Z}[t^{\pm 1}]/((6, 2(t + 2), t^2 + t + 1) + I') = \mathbb{Z}[t^{\pm 1}]/(2, t^2 + t + 1)$. By Corollary 4.2.6, $\mathbb{Z}[t^{\pm 1}]/(2, t^2 + t + 1)$ is a connected quandle. Therefore

$$\begin{aligned} &\mathbb{Z}[t^{\pm 1}]/(6, t^2 + t + 1) \\ &= \{0, 3, 3t, 3 + 3t\} \sqcup \{1 + 5t, 1 + 2t, 4 + 2t, 4 + 5t\} \sqcup \{2 + 4t, 2 + t, 5 + t, 5 + 4t\} \\ &\quad \sqcup \{1, 4, 1 + 3t, 4 + 3t\} \sqcup \{2 + 5t, 2 + 2t, 5 + 2t, 5 + 5t\} \sqcup \{3 + 4t, 3 + t, t, 4t\} \\ &\quad \sqcup \{2, 5, 2 + 3t, 5 + 3t\} \sqcup \{3 + 5t, 3 + 2t, 2t, 5t\} \sqcup \{4 + 4t, 4 + t, 1 + t, 1 + 4t\} \end{aligned}$$

is the maximal connected subquandle decomposition of $\mathbb{Z}[t^{\pm 1}]/(6, t^2 + t + 1)$, and we obtain that $\mathrm{depth}(\mathbb{Z}[t^{\pm 1}]/(6, t^2 + t + 1)) = 2$.

Proposition 4.4.4. *Let $n_0 \in \mathbb{Z}_{>0}$, $a \in \mathbb{Z}$ and put $n_{i+1} := n_i/\gcd(n_i, 1 + a)$ for any $i \in \mathbb{Z}_{\geq 0}$. Let l be the minimal number satisfying $n_l = n_{l+1}$. Then the Alexander quandle $\mathbb{Z}[t^{\pm 1}]/(n_0, t+a)$ is decomposed into N maximal connected subquandles, where $N = \prod_{i=0}^{l-1} \gcd(n_i, 1 + a)$, and any maximal connected subquandle of $\mathbb{Z}[t^{\pm 1}]/(n_0, t + a)$ is isomorphic to $\mathbb{Z}[t^{\pm 1}]/(n_l, t + a)$.*

Proof. By Theorem 4.2.4, for any $i \in \mathbb{Z}_{i\geq 0}$, $\mathbb{Z}[t^{\pm 1}]/(n_i, t + a)$ is decomposed into $\gcd(n_i, 1 + a)$ maximal connected subquandles, and any maximal connected subquandle of $\mathbb{Z}[t^{\pm 1}]/(n_i, t + a)$ is isomorphic to $\mathbb{Z}[t^{\pm 1}]/(n_{i+1}, t + a)$. Hence for any $i \in \mathbb{Z}_{>0}$, any element of $D^i(\{\mathbb{Z}[t^{\pm 1}]/(n_0, t + a)\})$ is isomorphic to $\mathbb{Z}[t^{\pm 1}]/(n_i, t + a)$, and we have $\#D^i(\{\mathbb{Z}[t^{\pm 1}]/(n_0, t + a)\}) = \prod_{j=0}^{i-1} \gcd(n_j, 1 + a)$. By Corollary 4.2.6, $n_k = n_{k+1}$, that is, $\gcd(n_k, 1 + a) = 1$ if and only if $D^k(\{\mathbb{Z}[t^{\pm 1}]/(n_0, t + a)\}) = D^{k+1}(\{\mathbb{Z}[t^{\pm 1}]/(n_0, t + a)\})$. Hence l is the minimal number satisfying $n_l = n_{l+1}$ if and only if $\mathrm{depth}(\mathbb{Z}[t^{\pm 1}]/(n_0, t + a)) = l$. By Theorem 4.3.2, $\mathbb{Z}[t^{\pm 1}]/(n_0, t + a) = \bigsqcup_{C \in D^l(\{\mathbb{Z}[t^{\pm 1}]/(n_0,t+a)\})} C$ is the maximal connected subquandle decomposition of $\mathbb{Z}[t^{\pm 1}]/(n_0, t + a)$. Since $N = \prod_{i=0}^{l-1} \gcd(n_i, 1 + a) = \#D^l(\{\mathbb{Z}[t^{\pm 1}]/(n_0, t + a)\})$, $\mathbb{Z}[t^{\pm 1}]/(n_0, t+a)$ is decomposed into N maximal connected subquandles, and any maximal connected subquandle of $\mathbb{Z}[t^{\pm 1}]/(n_0, t+a)$ is isomorphic to $\mathbb{Z}[t^{\pm 1}]/(n_l, t+a)$. $\square$

In Proposition 4.4.4, if $1+a$ is a prime number, $\mathbb{Z}[t^{\pm 1}]/(n_0, t+a)$ is decomposed into $|1+a|^l$ maximal connected subquandles, and any maximal connected subquandle of $\mathbb{Z}[t^{\pm 1}]/(n_0, t+a)$ is isomorphic to $\mathbb{Z}[t^{\pm 1}]/(k, t+a)$, where $n_0 = k(1+a)^l$ such that k and $1+a$ are relatively prime integers.

By Proposition 4.4.4, the following corollary holds.

Corollary 4.4.5. *For any $m \in \mathbb{Z}_{>0}$, the dihedral quandle R_m is decomposed into 2^l maximal connected subquandles, and any maximal connected subquandle of R_m is isomorphic to R_k, and* $\mathrm{depth}(R_m) = l$*, where k is an odd number, and $l \in \mathbb{Z}_{>0}$ such that $m = 2^l k$.*

Chapter 5

The Gordian distance and the unknotting number of handlebody-knots

The Gordian distance of two classical knots is the minimal number of crossing changes needed to be deformed each other. In particular, we call the Gordian distance of a classical knot and the trivial one the unknotting number of the classical knot. Clark, Elhamdadi, Saito and Yeatman [7] gave a lower bound for the Nakanishi index [41], which induced a lower bound for the unknotting number of classical knots. This is a generalization of the Przytycki's result [45]. In this chapter, we give lower bounds for the Gordian distance and the unknotting number of handlebody-knots.

Iwakiri [26] gave a lower bound for the unknotting number of handlebody-knots by using Alexander quandle colorings of its $\mathbb{Z}_2$ or $\mathbb{Z}_3$-flowed diagram. In this chapter, we extend the result in three directions. First, we extend from $\mathbb{Z}_2$, $\mathbb{Z}_3$-flows to any $\mathbb{Z}_m$-flow. Second, we extend from quandles to biquandles. Finally, we extend from the unknotting number to the Gordian distance. Thus we can determine the Gordian distance and the unknotting number of handlebody-knots more efficiently. We construct handlebody-knots with arbitrary Gordian distance and unknotting number and note that one of them can not be obtained by using Alexander quandle colorings introduced in [26].

This chapter is organized into four sections. In Section 5.1, we introduce the Gordian distance and the unknotting number of handlebody-knots and some properties of G-family of biquandles colorings. In Section 5.2, we show that there are linear relationships for Alexander biquandle colorings for any S^1-oriented handlebody-link. In Section 5.3, we give lower bounds for the Gordian distance and the unknotting number of handlebody-knots by using $\mathbb{Z}_m$-family of Alexander biquandles colorings. In Section 5.4, we construct handlebody-knots with Gordian distance n and unknotting number n for any $n \in \mathbb{Z}_{>0}$. Moreover, we note that one of them can not be obtained by using Alexander quandle colorings with $\mathbb{Z}_2, \mathbb{Z}_3$-flows introduced in [26].

5.1 The Gordian distance of handlebody-knots

A *crossing change* of an S^1-oriented handlebody-link H is that of a spatial trivalent graph representing H. This deformation can be realized by switching two handles depicted in Figure 5.1. It is easy to see that any two S^1-oriented handlebody-knots of the same genus can be related by a finite sequence of crossing changes. For any two S^1-oriented handlebody-knots H_1 and H_2 of the same genus, we define their *Gordian distance* $d(H_1, H_2)$ by the minimal number of crossing changes needed to be deformed each other. In particular, for any S^1-oriented handlebody-knot H and the S^1-oriented trivial handlebody-knot O of the same genus, we define $u(H) := d(H, O)$, which is called the *unknotting number* of H.

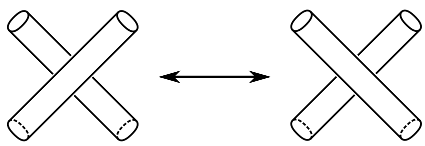


Figure 5.1: A crossing change of an S^1-oriented handlebody-link.

We remind a coloring for an S^1-oriented handlebody-link by a G-family of biquandles. Let G be a group, X be a G-family of biquandles and let (D, ρ) be a G-flowed diagram of an S^1-oriented handlebody-link H. An *X-coloring* of (D, ρ) is a map $C : \mathcal{SA}(D, \rho) \to X$ satisfying the conditions depicted in Figure 5.2 at each crossing and vertex. We denote by $\mathrm{Col}_X(D, \rho)$ the set of all X-colorings of (D, ρ). We note that $\mathrm{Col}_X(D, \rho)$ is a vector space over X when X is a $\mathbb{Z}_m$-family of Alexander biquandles and a field.

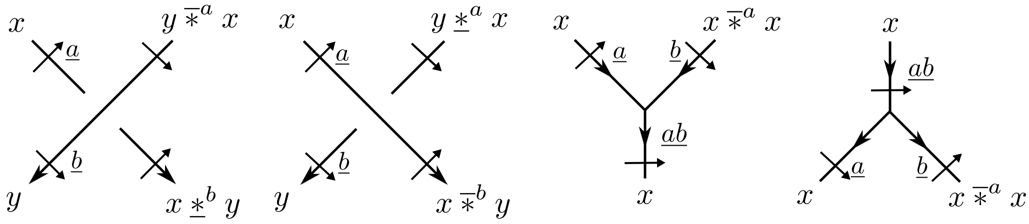


Figure 5.2: A G-family of biquandles coloring of (D, ρ).

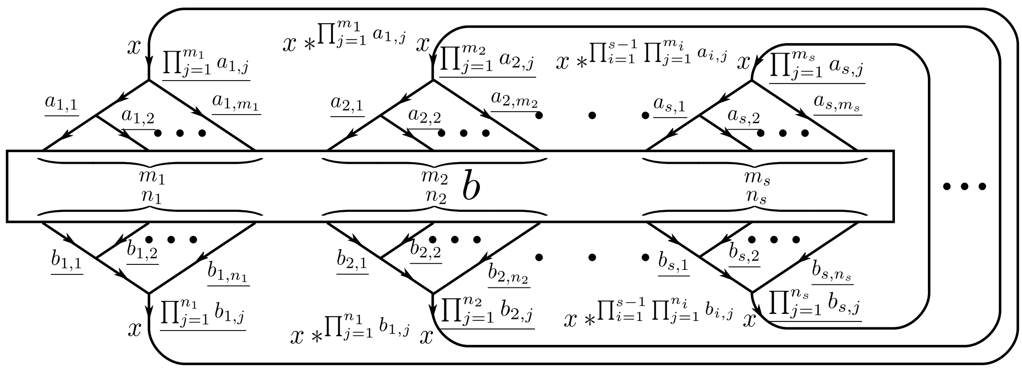
For any $m \in \mathbb{Z}_{\geq 0}$ and $\mathbb{Z}_m$-flow ρ of a diagram D of an S^1-oriented handlebody-link H, we define $\gcd \rho := \gcd\{\rho(a), m \mid a \in \mathcal{A}(D)\} \in \mathbb{Z}_{\geq 0}$, where we regard $\rho(a)$ as an arbitrary element of $\mathbb{Z}$ which represents $\rho(a) \in \mathbb{Z}_m$. Then we have the following lemma in the same way as in [20].

Lemma 5.1.1. *Let $m \in \mathbb{Z}_{\geq 0}$, (D, ρ) be a $\mathbb{Z}_m$-flowed diagram of an S^1-oriented handlebody-link H and let (D', ρ') be the associated $\mathbb{Z}_m$-flowed diagram of (D, ρ). Then it follows that* $\gcd \rho = \gcd \rho'$.

Proposition 5.1.2. *Let G be a group and let X be a G-family of biquandles. Then the following hold.*

1. *Let (D,ρ) be a G-flowed diagram of an S^1-oriented handlebody-link. Then it follows that $\#\mathrm{Col}_X(D,\rho) \geq \#X$.*
2. *Let (O,ψ) be a G-flowed diagram of an S^1-oriented m-component trivial handlebody-link. Then it follows that $\#\mathrm{Col}_X(O,\psi) = (\#X)^m$.*

Proof. 1. By Theorem 2.1.1 and [40], we can deform (D,ρ) into the G-flowed diagram (D',ρ') depicted in Figure 5.3 by a finite sequence of Reidemeinter moves preserving Y-orientations, where b is a classical l-braid, and $a_{i,1},\ldots,a_{i,m_i},b_{i,1},\ldots,b_{i,n_i} \in G$ for any $i=1,\ldots,s$. We note that $\prod_{j=1}^{m_i} a_{i,j} = \prod_{j=1}^{n_i} b_{i,j}$ for any $i=1,\ldots,s$, and $x \,\underline{*}^g\, x = x \,\overline{*}^g\, x$ for any $x \in X$ and $g \in G$. By Proposition 3.4.2, it is sufficient to prove that $\#\mathrm{Col}_X(D',\rho') \geq \#X$. Here for any $x \in X$ and $g \in G$, we write $x *^g x$ for $x \,\underline{*}^g\, x$ and $x \,\overline{*}^g\, x$ simply. Then for any $x \in X$, the assignment of elements of X to each semi-arc of (D',ρ') as shown in Figures 5.3 and 5.4 is an X-coloring, where each g_i represents an element of G in Figure 5.4. Therefore we have $\#\mathrm{Col}_X(D',\rho') \geq \#X$.



(D',ρ')

Figure 5.3: A G-flowed diagram (D',ρ') and its X-coloring.

2. It is sufficient to prove that $\#\mathrm{Col}_X(O,\psi) = \#X$ when $m=1$. Let (O_g,ψ_g) be a G-flowed diagram of an S^1-oriented trivial handlebody-knot of genus g. By Theorem 2.1.1, we can deform (O_g,ψ_g) into the G-flowed diagram (O'_g,ψ'_g) depicted in Figure 5.5 by a finite sequence of Reidemeinter moves preserving Y-orientations, where $a_i \in G$ for any $i=1,\ldots,g$, and e is the identity of G. By Proposition 3.4.2, it is sufficient to prove that $\#\mathrm{Col}_X(O'_g,\psi'_g) = \#X$. For any $x \in X$, the assignment of x to each semi-arc of (O'_g,ψ'_g) as shown in Figure 5.5 is an X-coloring. On the other hand, since any X-coloring of (O'_g,ψ'_g) is given by Figure 5.5 for some $x \in X$, we have $\#\mathrm{Col}_X(O'_g,\psi'_g) = \#X$.

□

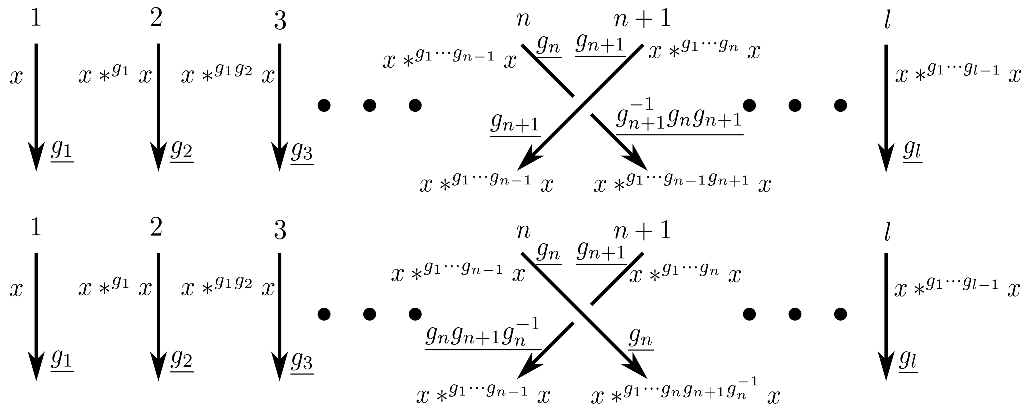


Figure 5.4: An X-coloring of (D', ρ') in the part of b.

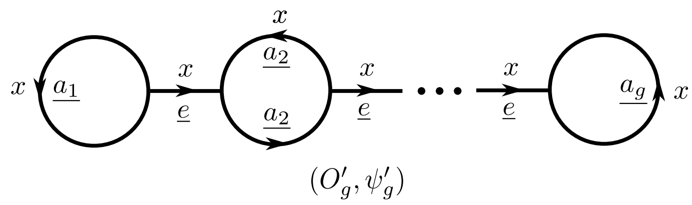


Figure 5.5: A G-flowed diagram (O'_g, ψ'_g) and its X-coloring.

5.2 Linear relationships for Alexander biquandle colorings

For any $\mathbb{Z}_m$-flowed diagram (D, ρ) of an S^1-oriented handlebody-link, we define the *Alexander numbering* of (D, ρ) by assigning elements of $\mathbb{Z}_m$ to each region of (D, ρ) as shown in Figure 5.6, where the unbounded region is labeled 0. It is an extension of the Alexander numbering of a classical knot diagram [1]. It is easy to see that for any $\mathbb{Z}_m$-flowed diagram (D, ρ) of an S^1-oriented handlebody-link, there uniquely exists the Alexander numbering of (D, ρ). For example, a $\mathbb{Z}_m$-flowed diagram of the handlebody-knot 5_2 [21] with the Alexander numbering is depicted in Figure 5.7. For any semi-arc α of (D, ρ), we denote by $\chi(\alpha)$ the Alexander number of the region which the normal orientation of α points to.

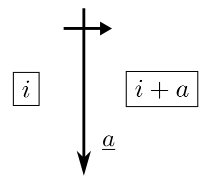


Figure 5.6: The Alexander numbering of (D, ρ).

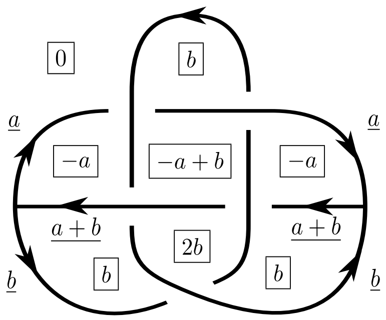


Figure 5.7: A $\mathbb{Z}_m$-flowed diagram of 5_2 with the Alexander numbering.

In the following, every component of a diagram of any S^1-oriented handlebody-link has a crossing at least 1. Let (D, ρ) be a $\mathbb{Z}_m$-flowed diagram of an S^1-oriented handlebody-link with the Alexander numbering and let X be a $\mathbb{Z}_m$-family of Alexander biquandles. We put $C(D, \rho) = \{c_1, \ldots, c_n\}$ and $V(D, \rho) = \{\tau_1, \ldots, \tau_{2k}\}$, where $C(D, \rho)$ and $V(D, \rho)$ are the set of all crossings of (D, ρ) and the one of all vertices of (D, ρ) respectively, where the sign of τ_i is 1 for any $i = 1, \ldots, k$ and -1 for any $i = k+1, \ldots, 2k$. Then we denote by x_i each semi-arc of (D, ρ) as shown in Figure 5.8, which implies $\mathcal{SA}(D, \rho) = \{x_1, \ldots, x_{2n+3k}\}$.

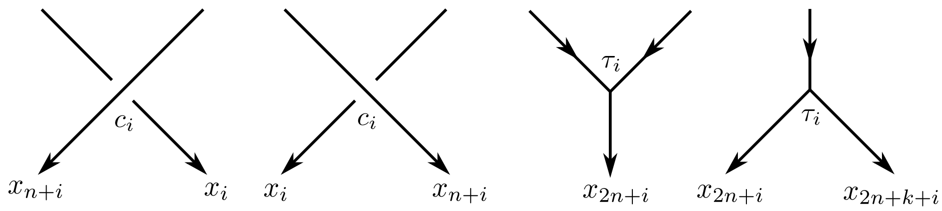


Figure 5.8: Semi-arcs x_i of (D, ρ).

We denote by u_i, v_i, v_i', w_i, α_i, β_i and γ_i the semi-arcs incident to a crossing c_i or a vertex τ_i as shown in Figure 5.9. We put $\rho_i := \rho(u_i) = \rho(w_i)$, $\psi_i := \rho(v_i) = \rho(v_i')$, $\eta_i := \rho(\alpha_i)$ and $\theta_i := \rho(\beta_i)$. We denote by $\epsilon_{c_i} \in \{\pm 1\}$ and $\epsilon_{\tau_i} \in \{\pm 1\}$ the signs of a crossing c_i and a vertex τ_i respectively (see Figure 5.9).

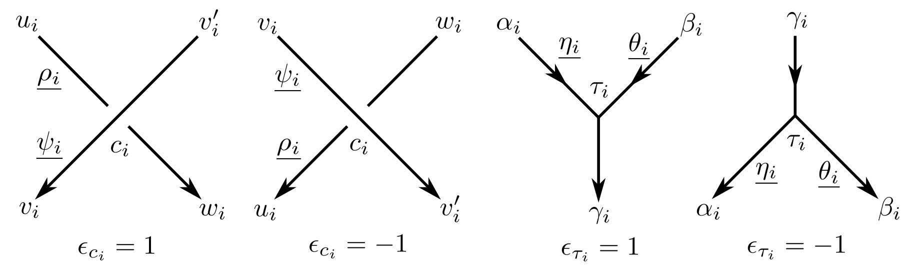


Figure 5.9: Notations.

For any semi-arcs $x, x' \in \mathcal{SA}(D, \rho)$, we put

$$\delta(x, x') := \begin{cases} 1 & (x = x'), \\ 0 & (x \neq x'). \end{cases}$$

We denote by $M(l, m; X)$ the set of $l \times m$ matrices over X. Then we define a matrix $A(D, \rho; X) = (a_{i,j}) \in M(2n + 4k, 2n + 3k; X)$ by

$$a_{i,j} = \begin{cases} \delta(u_i, x_j)t^{\psi_i} + \delta(v_i, x_j)(s^{\psi_i} - t^{\psi_i}) - \delta(w_i, x_j) & (1 \leq i \leq n), \\ -\delta(v_{i-n}, x_j)s^{\rho_{i-n}} + \delta(v'_{i-n}, x_j) & (n + 1 \leq i \leq 2n), \\ \delta(\alpha_{i-2n}, x_j) - \delta(\gamma_{i-2n}, x_j) & (2n + 1 \leq i \leq 2n + 2k), \\ \delta(\beta_{i-2n-2k}, x_j) - \delta(\gamma_{i-2n-2k}, x_j)s^{\eta_{i-2n-2k}} & (2n + 2k + 1 \leq i \leq 2n + 4k). \end{cases}$$

We note that $A(D, \rho; X)$ is determined up to permuting of rows and columns of the matrix. Then we can identify $\mathrm{Col}_X(D, \rho)$ with the set

$$\left\{ \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_{2n+3k} \end{pmatrix} \in X^{2n+3k} \;\middle|\; A(D, \rho; X) \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_{2n+3k} \end{pmatrix} = \mathbf{0} \right\}.$$

For example, let (E, ψ) be the $\mathbb{Z}_m$-flowed diagram of the handlebody-knot depicted in Figure 5.10. Then we have

$$A(E, \psi; X) = \begin{pmatrix} -1 & 0 & s^a - t^a & t^a & 0 & 0 & 0 \\ 0 & -1 & 0 & s^b - t^b & 0 & t^b & 0 \\ 0 & 1 & -s^b & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -s^a & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 0 & -s^a & 0 & 0 \\ 0 & 0 & 0 & 0 & -s^a & 0 & 1 \end{pmatrix}.$$

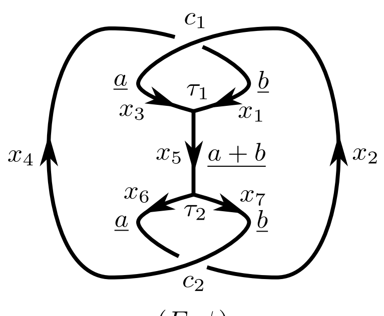


Figure 5.10: A $\mathbb{Z}_m$-flowed diagram (E, ψ).

Then we have the following proposition.

Proposition 5.2.1. *Let (D,ρ) be a $\mathbb{Z}_m$-flowed diagram of an S^1-oriented handlebody-link with the Alexander numbering and let X be a $\mathbb{Z}_m$-family of Alexander biquandles. Let $\boldsymbol{a}_i$ be the i-th row of $A(D,\rho;X)$. Then it follows that*

$$\begin{aligned}&\sum_{i=1}^{n}\epsilon_{c_i}t^{-\chi(w_i)}(s^{\rho_i}-t^{\rho_i})\boldsymbol{a}_i+\sum_{i=1}^{n}\epsilon_{c_i}t^{-\chi(v_i')}(s^{\psi_i}-t^{\psi_i})\boldsymbol{a}_{n+i}\\&+\sum_{i=1}^{2k}\epsilon_{\tau_i}t^{-\chi(\alpha_i)}(s^{\eta_i}-t^{\eta_i})\boldsymbol{a}_{2n+i}+\sum_{i=1}^{2k}\epsilon_{\tau_i}t^{-\chi(\beta_i)}(s^{\theta_i}-t^{\theta_i})\boldsymbol{a}_{2n+2k+i}=\mathbf{0}.\end{aligned}$$

Proof. For any semi-arc x incident to a crossing or a vertex σ, we put

$$\epsilon(x;\sigma):=\begin{cases}1 & \text{if the orientation of } x \text{ points to } \sigma,\\ -1 & \text{otherwise.}\end{cases}$$

We set $(a_{i,j}):=A(D,\rho;X)$. It is sufficient to prove that for any $j=1,2,\ldots,2n+3k$,

$$\begin{aligned}&\sum_{i=1}^{n}\epsilon_{c_i}t^{-\chi(w_i)}(s^{\rho_i}-t^{\rho_i})a_{i,j}+\sum_{i=1}^{n}\epsilon_{c_i}t^{-\chi(v_i')}(s^{\psi_i}-t^{\psi_i})a_{n+i,j}\\&+\sum_{i=1}^{2k}\epsilon_{\tau_i}t^{-\chi(\alpha_i)}(s^{\eta_i}-t^{\eta_i})a_{2n+i,j}+\sum_{i=1}^{2k}\epsilon_{\tau_i}t^{-\chi(\beta_i)}(s^{\theta_i}-t^{\theta_i})a_{2n+2k+i,j}=0.\end{aligned}$$

For the first term, we have

$$\begin{aligned}&\epsilon_{c_i}t^{-\chi(w_i)}(s^{\rho_i}-t^{\rho_i})\delta(u_i,x_j)t^{\psi_i}=\delta(u_i,x_j)\epsilon(u_i;c_i)t^{-\chi(u_i)}(s^{\rho(u_i)}-t^{\rho(u_i)}),\\&\epsilon_{c_i}t^{-\chi(w_i)}(s^{\rho_i}-t^{\rho_i})\delta(v_i,x_j)(s^{\psi_i}-t^{\psi_i})\\&=\epsilon_{c_i}t^{-\chi(w_i)}s^{\rho_i}\delta(v_i,x_j)(s^{\psi_i}-t^{\psi_i})-\epsilon_{c_i}t^{-\chi(w_i)}t^{\rho_i}\delta(v_i,x_j)(s^{\psi_i}-t^{\psi_i})\\&=\epsilon_{c_i}t^{-\chi(w_i)}\delta(v_i,x_j)(s^{\psi_i}-t^{\psi_i})s^{\rho_i}+\delta(v_i,x_j)\epsilon(v_i;c_i)t^{-\chi(v_i)}(s^{\rho(v_i)}-t^{\rho(v_i)}),\\&\epsilon_{c_i}t^{-\chi(w_i)}(s^{\rho_i}-t^{\rho_i})(-\delta(w_i,x_j))=\delta(w_i,x_j)\epsilon(w_i;c_i)t^{-\chi(w_i)}(s^{\rho(w_i)}-t^{\rho(w_i)}).\end{aligned}\tag{1}$$

For the second term, we have

$$\begin{aligned}&\epsilon_{c_i}t^{-\chi(v_i')}(s^{\psi_i}-t^{\psi_i})(-\delta(v_i,x_j)s^{\rho_i})=-\epsilon_{c_i}t^{-\chi(v_i')}\delta(v_i,x_j)(s^{\psi_i}-t^{\psi_i})s^{\rho_i},\\&\epsilon_{c_i}t^{-\chi(v_i')}(s^{\psi_i}-t^{\psi_i})\delta(v_i',x_j)=\delta(v_i',x_j)\epsilon(v_i';c_i)t^{-\chi(v_i')}(s^{\rho(v_i')}-t^{\rho(v_i')}).\end{aligned}\tag{2}$$

For the third term, we have

$$\begin{aligned}&\epsilon_{\tau_i}t^{-\chi(\alpha_i)}(s^{\eta_i}-t^{\eta_i})\delta(\alpha_i,x_j)=\delta(\alpha_i,x_j)\epsilon(\alpha_i;\tau_i)t^{-\chi(\alpha_i)}(s^{\rho(\alpha_i)}-t^{\rho(\alpha_i)}),\\&\epsilon_{\tau_i}t^{-\chi(\alpha_i)}(s^{\eta_i}-t^{\eta_i})(-\delta(\gamma_i,x_j))=\delta(\gamma_i,x_j)\epsilon(\gamma_i;\tau_i)t^{-\chi(\gamma_i)}t^{\theta_i}(s^{\eta_i}-t^{\eta_i}).\end{aligned}\tag{3}$$

For the last term, we have

$$\begin{aligned}&\epsilon_{\tau_i}t^{-\chi(\beta_i)}(s^{\theta_i}-t^{\theta_i})\delta(\beta_i,x_j)=\delta(\beta_i,x_j)\epsilon(\beta_i;\tau_i)t^{-\chi(\beta_i)}(s^{\rho(\beta_i)}-t^{\rho(\beta_i)}),\\&\epsilon_{\tau_i}t^{-\chi(\beta_i)}(s^{\theta_i}-t^{\theta_i})(-\delta(\gamma_i,x_j)s^{\eta_i})=\delta(\gamma_i,x_j)\epsilon(\gamma_i;\tau_i)t^{-\chi(\gamma_i)}(s^{\theta_i}-t^{\theta_i})s^{\eta_i}.\end{aligned}\tag{4}$$

We note that

$$\begin{aligned}(1)+(2)&=\delta(v_i,x_j)\epsilon(v_i;c_i)t^{-\chi(v_i)}(s^{\rho(v_i)}-t^{\rho(v_i)}),\\(3)+(4)&=\delta(\gamma_i,x_j)\epsilon(\gamma_i;\tau_i)t^{-\chi(\gamma_i)}(s^{\rho(\gamma_i)}-t^{\rho(\gamma_i)}).\end{aligned}$$

Therefore for any $j=1,2,\ldots,2n+3k$, it follows that

$$\begin{aligned}&\sum_{i=1}^{n}\epsilon_{c_i}t^{-\chi(w_i)}(s^{\rho_i}-t^{\rho_i})a_{i,j}+\sum_{i=1}^{n}\epsilon_{c_i}t^{-\chi(v_i')}(s^{\psi_i}-t^{\psi_i})a_{n+i,j}\\&\quad+\sum_{i=1}^{2k}\epsilon_{\tau_i}t^{-\chi(\alpha_i)}(s^{\eta_i}-t^{\eta_i})a_{2n+i,j}+\sum_{i=1}^{2k}\epsilon_{\tau_i}t^{-\chi(\beta_i)}(s^{\theta_i}-t^{\theta_i})a_{2n+2k+i,j}\\&=\sum_{i=1}^{n}(\delta(u_i,x_j)\epsilon(u_i;c_i)t^{-\chi(u_i)}(s^{\rho(u_i)}-t^{\rho(u_i)})\\&\qquad+\delta(v_i,x_j)\epsilon(v_i;c_i)t^{-\chi(v_i)}(s^{\rho(v_i)}-t^{\rho(v_i)})\\&\qquad+\delta(v_i',x_j)\epsilon(v_i';c_i)t^{-\chi(v_i')}(s^{\rho(v_i')}-t^{\rho(v_i')})\\&\qquad+\delta(w_i,x_j)\epsilon(w_i;c_i)t^{-\chi(w_i)}(s^{\rho(w_i)}-t^{\rho(w_i)}))\\&\quad+\sum_{i=1}^{2k}(\delta(\alpha_i,x_j)\epsilon(\alpha_i;\tau_i)t^{-\chi(\alpha_i)}(s^{\rho(\alpha_i)}-t^{\rho(\alpha_i)})\\&\qquad+\delta(\beta_i,x_j)\epsilon(\beta_i;\tau_i)t^{-\chi(\beta_i)}(s^{\rho(\beta_i)}-t^{\rho(\beta_i)})\\&\qquad+\delta(\gamma_i,x_j)\epsilon(\gamma_i;\tau_i)t^{-\chi(\gamma_i)}(s^{\rho(\gamma_i)}-t^{\rho(\gamma_i)}))\\&=t^{-\chi(x_j)}(s^{\rho(x_j)}-t^{\rho(x_j)})-t^{-\chi(x_j)}(s^{\rho(x_j)}-t^{\rho(x_j)})\\&=0.\end{aligned}$$

□

Let X be an Alexander biquandle and let $m=\operatorname{type}X$. Then X is also a $\mathbb{Z}_m$-family of Alexander biquandles. Let D be an oriented classical link diagram. We can regard D as a $\mathbb{Z}_m$-flowed diagram $(D,\rho_{(1)})$ of an S^1-oriented handlebody-link whose components are of genus 1, where $\rho_{(1)}$ is the constant map to 1. Hence we can regard an X-coloring of D as an X-coloring of $(D,\rho_{(1)})$. We define a matrix $A(D;X)\in M(2n,2n;X)$ by $A(D;X)=A(D,\rho_{(1)};X)$, where n is the number of crossings of D. Then the set of all X-colorings of D, denoted $\mathrm{Col}_X(D)$, is given by

$$\mathrm{Col}_X(D)=\left\{\begin{pmatrix}z_1\\z_2\\\vdots\\z_{2n}\end{pmatrix}\in X^{2n}\,\middle|\,A(D;X)\begin{pmatrix}z_1\\z_2\\\vdots\\z_{2n}\end{pmatrix}=\mathbf{0}\right\}.$$

Therefore we obtain the following corollary.

Corollary 5.2.2. *Let D be a diagram of an oriented classical link with the Alexander numbering and let X be an Alexander biquandle. Let $\boldsymbol{a}_i$ be the i-th row of $A(D;X)$.*

Then it follows that

$$\sum_{i=1}^{n} \epsilon_{c_i} t^{-\chi(w_i)}(s-t)\boldsymbol{a}_i + \sum_{i=1}^{n} \epsilon_{c_i} t^{-\chi(v_i')}(s-t)\boldsymbol{a}_{n+i} = \boldsymbol{0}.$$

5.3 Results

In this section, we give lower bounds for the Gordian distance and the unknotting number of S^1-oriented handlebody-knots.

Theorem 5.3.1. *Let H_i be an S^1-oriented handlebody-knot of genus g and let D_i be a diagram of H_i $(i = 1, 2)$. Let $X = \mathbb{Z}_p[t^{\pm 1}]/(f(t))$ which is a $\mathbb{Z}_m$-family of Alexander biquandles, where p is a prime number, $s \in \mathbb{Z}_p[t^{\pm 1}]$ and $f(t) \in \mathbb{Z}_p[t^{\pm 1}]$ is an irreducible polynomial. Then it follows that*

$$\max_{\rho_1 \in \mathrm{Flow}(D_1;\mathbb{Z}_m)} \min_{\substack{\rho_2 \in \mathrm{Flow}(D_2;\mathbb{Z}_m) \\ \gcd \rho_1 = \gcd \rho_2}} |\dim \mathrm{Col}_X(D_1, \rho_1) - \dim \mathrm{Col}_X(D_2, \rho_2)| \leq d(H_1, H_2).$$

Proof. Let (D, ρ) be a $\mathbb{Z}_m$-flowed diagram of an S^1-oriented handlebody-knot and let $C(D, \rho) = \{c_1, \ldots, c_n\}$ and $V(D, \rho) = \{\tau_1, \ldots, \tau_{2k}\}$. Let $(\overline{D}, \overline{\rho})$ be the $\mathbb{Z}_m$-flowed diagram of an S^1-oriented handlebody-knot which is obtained from (D, ρ) by the crossing change at c_1 and let $C(\overline{D}, \overline{\rho}) = \{\overline{c}_1, \ldots, \overline{c}_n\}$ and $V(\overline{D}, \overline{\rho}) = \{\overline{\tau}_1, \ldots, \overline{\tau}_{2k}\}$, where $\overline{\rho}$, $\overline{c}_i$ and $\overline{\tau}_i$ originate from ρ, c_i and τ_i naturally and respectively (see Figure 5.11). In the following, we show that

$$|\dim \mathrm{Col}_X(D, \rho) - \dim \mathrm{Col}_X(\overline{D}, \overline{\rho})| \leq 1,$$

that is,

$$|\operatorname{rank} A(D, \rho; X) - \operatorname{rank} A(\overline{D}, \overline{\rho}; X)| \leq 1.$$

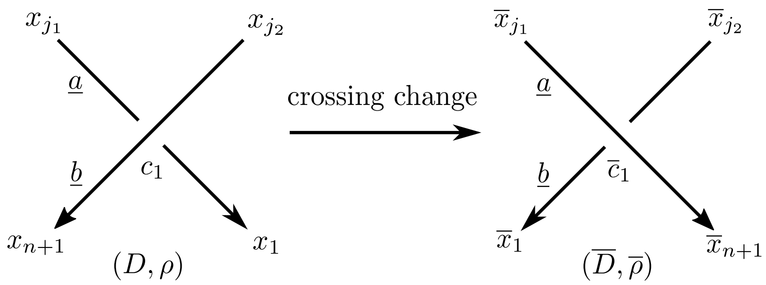


Figure 5.11: The crossing change at c_1.

We may assume that c_1 is a positive crossing and $\overline{c}_1$ is a negative crossing. We denote by $\overline{x}_i$ each semi-arc of $(\overline{D}, \overline{\rho})$ in the same way as in Figure 5.8 with respect to $\overline{c}_i$ or $\overline{\tau}_i$, and so are $\overline{v}_i'$, $\overline{w}_i$, $\overline{\alpha}_i$, $\overline{\beta}_i$, $\overline{\rho}_i$, $\overline{\psi}_i$, $\overline{\eta}_i$, $\overline{\theta}_i$, $\overline{\epsilon}_{c_i}$ and $\overline{\epsilon}_{\tau_i}$ (see Figure 5.9). We denote by x_{j_1} and x_{j_2} the semi-arcs which point to the crossing c_1 of (D, ρ) as shown in Figure 5.11, and we put $a := \rho_1 = \overline{\psi}_1$ and $b := \psi_1 = \overline{\rho}_1$. We remind that $\mathrm{Col}_X(D, \rho)$ and

$\mathrm{Col}_X(\overline{D},\overline{\rho})$ are vector spaces over X since X is a $\mathbb{Z}_m$-family of Alexander biquandles and a field.

Let $\boldsymbol{a}_i$, $\overline{\boldsymbol{a}}_i$ and $\hat{\boldsymbol{a}}_i$ be the i-th rows of $A(D,\rho;X)$, $A(\overline{D},\overline{\rho};X)$ and $\hat{A}(\overline{D},\overline{\rho};X)$ respectively, where $\hat{A}(\overline{D},\overline{\rho};X)$ is the matrix obtained by permuting the first column and the $(n+1)$-th column of $A(\overline{D},\overline{\rho};X)$. We put $(a_{i,j}) := A(D,\rho;X)$, $(\overline{a}_{i,j}) := A(\overline{D},\overline{\rho};X)$ and $(\hat{a}_{i,j}) := \hat{A}(\overline{D},\overline{\rho};X)$. Then we have $\boldsymbol{a}_i = \hat{\boldsymbol{a}}_i$ when $i \neq 1, n+1$. We note that $\operatorname{rank} A(\overline{D},\overline{\rho};X) = \operatorname{rank} \hat{A}(\overline{D},\overline{\rho};X)$ and

$$\begin{aligned}
\boldsymbol{a}_1 &= (-1,0,\dots,0,\overset{\overset{j_1}{\vee}}{t^b},0,\dots,0,\overset{\overset{n+1}{\vee}}{s^b-t^b},0,\dots,0),\\
\boldsymbol{a}_{n+1} &= (0,\dots,0,\overset{\overset{j_2}{\vee}}{1},0,\dots,0,\overset{\overset{n+1}{\vee}}{-s^a},0,\dots,0),\\
\overline{\boldsymbol{a}}_1 &= (t^a,0,\dots,0,\overset{\overset{j_1}{\vee}}{s^a-t^a},0,\dots,0,\overset{\overset{j_2}{\vee}}{-1},0,\dots,0),\\
\overline{\boldsymbol{a}}_{n+1} &= (0,\dots,0,\overset{\overset{j_1}{\vee}}{-s^b},0,\dots,0,\overset{\overset{n+1}{\vee}}{1},0,\dots,0),\\
\hat{\boldsymbol{a}}_1 &= (0,\dots,0,\overset{\overset{j_1}{\vee}}{s^a-t^a},0,\dots,0,\overset{\overset{j_2}{\vee}}{-1},0,\dots,0,\overset{\overset{n+1}{\vee}}{t^a},0,\dots,0),\\
\hat{\boldsymbol{a}}_{n+1} &= (1,0,\dots,0,\overset{\overset{j_1}{\vee}}{-s^b},0,\dots,0).
\end{aligned}$$

By Proposition 5.2.1, we obtain

$$\begin{aligned}
&\sum_{i=1}^{n}\epsilon_{c_i}t^{-\chi(w_i)}(s^{\rho_i}-t^{\rho_i})\boldsymbol{a}_i + \sum_{i=1}^{n}\epsilon_{c_i}t^{-\chi(v_i')}(s^{\psi_i}-t^{\psi_i})\boldsymbol{a}_{n+i}\\
&\quad+\sum_{i=1}^{2k}\epsilon_{\tau_i}t^{-\chi(\alpha_i)}(s^{\eta_i}-t^{\eta_i})\boldsymbol{a}_{2n+i} + \sum_{i=1}^{2k}\epsilon_{\tau_i}t^{-\chi(\beta_i)}(s^{\theta_i}-t^{\theta_i})\boldsymbol{a}_{2n+2k+i} = \boldsymbol{0}
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{i=1}^{n}\overline{\epsilon}_{c_i}t^{-\chi(\overline{w}_i)}(s^{\overline{\rho}_i}-t^{\overline{\rho}_i})\overline{\boldsymbol{a}}_i + \sum_{i=1}^{n}\overline{\epsilon}_{c_i}t^{-\chi(\overline{v}_i')}(s^{\overline{\psi}_i}-t^{\overline{\psi}_i})\overline{\boldsymbol{a}}_{n+i}\\
&\quad+\sum_{i=1}^{2k}\overline{\epsilon}_{\tau_i}t^{-\chi(\overline{\alpha}_i)}(s^{\overline{\eta}_i}-t^{\overline{\eta}_i})\overline{\boldsymbol{a}}_{2n+i} + \sum_{i=1}^{2k}\overline{\epsilon}_{\tau_i}t^{-\chi(\overline{\beta}_i)}(s^{\overline{\theta}_i}-t^{\overline{\theta}_i})\overline{\boldsymbol{a}}_{2n+2k+i}\\
&=\sum_{i=1}^{n}\overline{\epsilon}_{c_i}t^{-\chi(\overline{w}_i)}(s^{\overline{\rho}_i}-t^{\overline{\rho}_i})\hat{\boldsymbol{a}}_i + \sum_{i=1}^{n}\overline{\epsilon}_{c_i}t^{-\chi(\overline{v}_i')}(s^{\overline{\psi}_i}-t^{\overline{\psi}_i})\hat{\boldsymbol{a}}_{n+i}\\
&\quad+\sum_{i=1}^{2k}\overline{\epsilon}_{\tau_i}t^{-\chi(\overline{\alpha}_i)}(s^{\overline{\eta}_i}-t^{\overline{\eta}_i})\hat{\boldsymbol{a}}_{2n+i} + \sum_{i=1}^{2k}\overline{\epsilon}_{\tau_i}t^{-\chi(\overline{\beta}_i)}(s^{\overline{\theta}_i}-t^{\overline{\theta}_i})\hat{\boldsymbol{a}}_{2n+2k+i} = \boldsymbol{0}.
\end{aligned}$$

If $\epsilon_{c_1}t^{-\chi(w_1)}(s^{\rho_1}-t^{\rho_1}) = 0$, we have $s^{\rho_1}-t^{\rho_1} = s^a - t^a = 0$, which implies that $\boldsymbol{a}_{n+1} = -\hat{\boldsymbol{a}}_1$. Hence it follows that

$$|\operatorname{rank} A(D,\rho;X) - \operatorname{rank} A(\overline{D},\overline{\rho};X)| = |\operatorname{rank} A(D,\rho;X) - \operatorname{rank} \hat{A}(\overline{D},\overline{\rho};X)| \leq 1.$$

If $\overline{\epsilon}_{c_1} t^{-\chi(\overline{w}_1)}(s^{\overline{\rho}_1} - t^{\overline{\rho}_1}) = 0$, we have $s^{\overline{\rho}_1} - t^{\overline{\rho}_1} = s^b - t^b = 0$, which implies that $\boldsymbol{a}_1 = -\hat{\boldsymbol{a}}_{n+1}$. Hence it follows that

$$|\operatorname{rank} A(D,\rho;X) - \operatorname{rank} A(\overline{D},\overline{\rho};X)| = |\operatorname{rank} A(D,\rho;X) - \operatorname{rank} \hat{A}(\overline{D},\overline{\rho};X)| \leq 1.$$

If $\epsilon_{c_1} t^{-\chi(w_1)}(s^{\rho_1} - t^{\rho_1}) \neq 0$ and $\overline{\epsilon}_{c_1} t^{-\chi(\overline{w}_1)}(s^{\overline{\rho}_1} - t^{\overline{\rho}_1}) \neq 0$, we can represent $\boldsymbol{a}_1$ and $\overline{\boldsymbol{a}}_1$ as linear combinations of $\boldsymbol{a}_2, \ldots, \boldsymbol{a}_{2n+4k}$ and $\overline{\boldsymbol{a}}_2, \ldots, \overline{\boldsymbol{a}}_{2n+4k}$ respectively. Hence it follows that

$$\operatorname{rank} A(D,\rho;X) = \operatorname{rank} \begin{pmatrix} \boldsymbol{a}_2 \\ \vdots \\ \boldsymbol{a}_{2n+4k} \end{pmatrix}, \quad \operatorname{rank} A(\overline{D},\overline{\rho};X) = \operatorname{rank} \begin{pmatrix} \overline{\boldsymbol{a}}_2 \\ \vdots \\ \overline{\boldsymbol{a}}_{2n+4k} \end{pmatrix},$$

which implies that

$$\begin{aligned} |\operatorname{rank} A(D,\rho;X) - \operatorname{rank} A(\overline{D},\overline{\rho};X)| &= \left| \operatorname{rank} \begin{pmatrix} \boldsymbol{a}_2 \\ \vdots \\ \boldsymbol{a}_{2n+4k} \end{pmatrix} - \operatorname{rank} \begin{pmatrix} \overline{\boldsymbol{a}}_2 \\ \vdots \\ \overline{\boldsymbol{a}}_{2n+4k} \end{pmatrix} \right| \\ &= \left| \operatorname{rank} \begin{pmatrix} \boldsymbol{a}_2 \\ \vdots \\ \boldsymbol{a}_{2n+4k} \end{pmatrix} - \operatorname{rank} \begin{pmatrix} \hat{\boldsymbol{a}}_2 \\ \vdots \\ \hat{\boldsymbol{a}}_{2n+4k} \end{pmatrix} \right| \\ &\leq 1. \end{aligned}$$

Consequently, if we can deform H_1 into H_2 by crossing changes at l crossings, then for any $\mathbb{Z}_m$-flowed diagram (D_1,ρ_1) of H_1, there exists a $\mathbb{Z}_m$-flowed diagram (D_2,ρ_2) of H_2 satisfying $\gcd \rho_1 = \gcd \rho_2$ and

$$|\dim \operatorname{Col}_X(D_1,\rho_1) - \dim \operatorname{Col}_X(D_2,\rho_2)| \leq l$$

by Lemma 5.1.1. Therefore it follows that

$$\max_{\rho_1 \in \operatorname{Flow}(D_1;\mathbb{Z}_m)} \min_{\substack{\rho_2 \in \operatorname{Flow}(D_2;\mathbb{Z}_m) \\ \gcd \rho_1 = \gcd \rho_2}} |\dim \operatorname{Col}_X(D_1,\rho_1) - \dim \operatorname{Col}_X(D_2,\rho_2)| \leq d(H_1,H_2).$$

□

By Proposition 5.1.2 and Theorem 5.3.1, the following corollary holds immediately.

Corollary 5.3.2. *Let H be an S^1-oriented handlebody-knot and let D be a diagram of H. Let $X = \mathbb{Z}_p[t^{\pm 1}]/(f(t))$ which is a $\mathbb{Z}_m$-family of Alexander biquandles, where p is a prime number, $s \in \mathbb{Z}_p[t^{\pm 1}]$ and $f(t) \in \mathbb{Z}_p[t^{\pm 1}]$ is an irreducible polynomial. Then it follows that*

$$\max_{\rho \in \operatorname{Flow}(D;\mathbb{Z}_m)} \dim \operatorname{Col}_X(D,\rho) - 1 \leq u(H).$$

5.4 Examples

In this section, we give some examples. In Example 5.4.1, we give a handlebody-knot with unknotting number 2, and in Remark 5.4.2, we note that it can not be obtained by using Alexander quandle colorings with $\mathbb{Z}_2, \mathbb{Z}_3$-flows introduced in [26]. In Example 5.4.3, we give three handlebody-knots with unknotting number n for any $n \in \mathbb{Z}_{>0}$. In Example 5.4.4, we give two handlebody-knots with their Gordian distance n for any $n \in \mathbb{Z}_{>0}$.

Example 5.4.1. Let H be the handlebody-knot represented by the $\mathbb{Z}_{10}$-flowed diagram (D, ρ) depicted in Figure 5.12. Then we show that $u(H) = 2$.

Let $s = 1 \in \mathbb{Z}_3[t^{\pm 1}]$ and let $f(t) = t^4 + 2t^3 + t^2 + 2t + 1 \in \mathbb{Z}_3[t^{\pm 1}]$, which is an irreducible polynomial. Then $X := \mathbb{Z}_3[t^{\pm 1}]/(f(t))$ is a $\mathbb{Z}_{10}$-family of Alexander biquandles. Then for any $x, y, z \in X$, the assignment of them to each semi-arc of (D, ρ) as shown in Figure 5.12 is an X-coloring of (D, ρ), which implies $\dim \mathrm{Col}_X(D, \rho) \geq 3$. By Corollary 5.3.2, we obtain $2 \leq u(H)$. On the other hand, we can deform H into a trivial handlebody-knot by the crossing changes at two crossings surrounded by dotted circles depicted in Figure 5.12. Therefore it follows that $u(H) = 2$.

(D, ρ) :

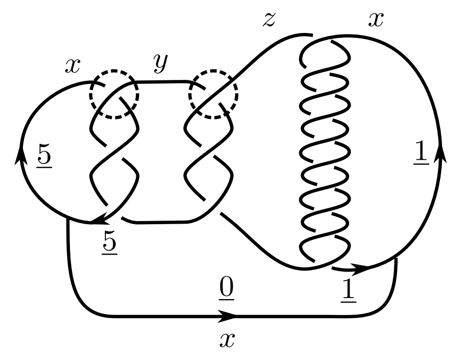


Figure 5.12: A $\mathbb{Z}_{10}$-flowed diagram (D, ρ) of H.

Remark 5.4.2. We show that the result in Example 5.4.1 can not be obtained by using Alexander quandle colorings with $\mathbb{Z}_2, \mathbb{Z}_3$-flows introduced in [26].

Let H be the handlebody-knot represented by the $\mathbb{Z}_m$-flowed diagram $(D, \rho(a, b))$ depicted in Figure 5.13 for any $m = 2, 3$ and $a, b \in \mathbb{Z}_m$. Let p be a prime number, $s = 1 \in \mathbb{Z}_p[t^{\pm 1}]$, $f(t)$ be an irreducible polynomial in $\mathbb{Z}_p[t^{\pm 1}]$ and let $X = \mathbb{Z}_p[t^{\pm 1}]/(f(t))$ which is a $\mathbb{Z}_m$-family of Alexander (bi)quandles. We note that $\mathrm{Col}_X(D, \rho(a, b))$ is generated by $x, y, z \in X$ as shown in Figure 5.13 for any $m = 2, 3$ and $a, b \in \mathbb{Z}_m$. If $(a, b) = (1, 0)$, x, y and z need to satisfy the following relations:

$$\begin{aligned}
&(t^2 - t + 1)x - (t^2 - t + 1)y = 0, \\
&-t(t^2 - t + 1)x + t^{-1}(t + 1)(t - 1)(t^2 - t + 1)y + t^{-1}(t^2 - t + 1)z = 0, \\
&-t^{-1}(t - 1)(t^2 - t + 1)x + t^{-2}(t^2 - t - 1)(t^2 - t + 1)y + t^{-2}(t^2 - t + 1)z = 0, \\
&((t^3 + t^2 - 1)(t^2 - t + 1) - t)x - ((t^3 + t^2 - 1)(t^2 - t + 1) - t)z = 0,
\end{aligned}$$

that is,

$$M \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

where

$$M = \begin{pmatrix} t^2 - t + 1 & -(t^2 - t + 1) & 0 \\ -t(t^2 - t + 1) & t^{-1}(t+1)(t-1)(t^2 - t + 1) & t^{-1}(t^2 - t + 1) \\ -t^{-1}(t-1)(t^2 - t + 1) & t^{-2}(t^2 - t - 1)(t^2 - t + 1) & t^{-2}(t^2 - t + 1) \\ (t^3 + t^2 - 1)(t^2 - t + 1) - t & 0 & -(t^3 + t^2 - 1)(t^2 - t + 1) + t \end{pmatrix}.$$

These relations are obtained from crossings c_1, c_2, c_3 and c_4 as shown in Figure 5.13. When $t^2 - t + 1 \neq 0$ in X, it is clearly that $\operatorname{rank} M \geq 1$. When $t^2 - t + 1 = 0$ in X, we have

$$M = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -t & 0 & t \end{pmatrix},$$

which implies that $\operatorname{rank} M = 1$. Hence we have $\dim \operatorname{Col}_X(D, \rho(1,0)) = 3 - \operatorname{rank} M \leq 2$. Therefore we can not obtain $2 \leq u(H)$.

We can prove the remaining cases in the same way.

$(D, \rho(a, b))$:

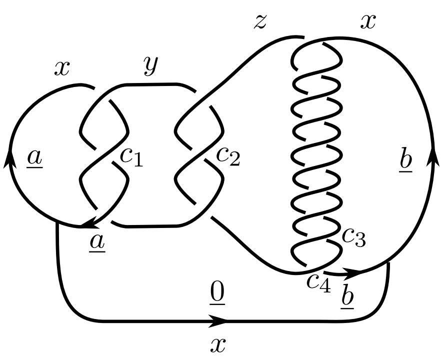


Figure 5.13: A $\mathbb{Z}_m$-flowed diagram $(D, \rho(a, b))$ of H.

Example 5.4.3. Let A_n, B_n and C_n be the handlebody-knots represented by the $\mathbb{Z}_8$-flowed diagram (D_{A_n}, ρ_{A_n}), the $\mathbb{Z}_{24}$-flowed diagram (D_{B_n}, ρ_{B_n}) and the $\mathbb{Z}_8$-flowed diagram (D_{C_n}, ρ_{C_n}) depicted in Figure 5.14, 5.15 and 5.16 respectively for any $n \in \mathbb{Z}_{>0}$. Then we show that $u(A_n) = u(B_n) = u(C_n) = n$.

1. Let $s = t + 1 \in \mathbb{Z}_3[t^{\pm 1}]$ and let $f(t) = t^2 + t + 2 \in \mathbb{Z}_3[t^{\pm 1}]$, which is an irreducible polynomial. Then $X := \mathbb{Z}_3[t^{\pm 1}]/(f(t))$ is a $\mathbb{Z}_8$-family of Alexander biquandles. Then for any $x_0, x_1, \ldots, x_n \in X$, the assignment of them to each semi-arc of

(D_{A_n}, ρ_{A_n}) as shown in Figure 5.14 is an X-coloring of (D_{A_n}, ρ_{A_n}), which implies $\dim \mathrm{Col}_X(D_{A_n}, \rho_{A_n}) \geq n+1$. By Corollary 5.3.2, we obtain $n \leq u(A_n)$. On the other hand, we can deform A_n into a trivial handlebody-knot by the crossing changes at n crossings surrounded by dotted circles depicted in Figure 5.14. Therefore it follows that $u(A_n) = n$.

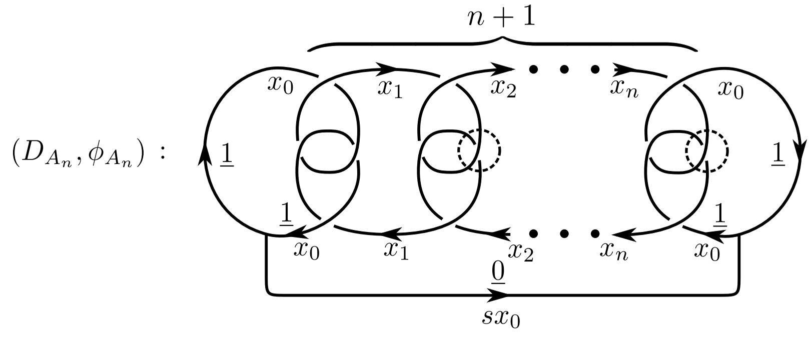


Figure 5.14: A $\mathbb{Z}_8$-flowed diagram (D_{A_n}, ρ_{A_n}) of A_n.

2. Let $s = t^2 + 1 \in \mathbb{Z}_5[t^{\pm 1}]$ and let $f(t) = t^2 + 2t + 4 \in \mathbb{Z}_5[t^{\pm 1}]$, which is an irreducible polynomial. Then $X := \mathbb{Z}_5[t^{\pm 1}]/(f(t))$ is a $\mathbb{Z}_{24}$-family of Alexander biquandles. Then for any $x_0, x_1, \ldots, x_n \in X$, the assignment of them to each semi-arc of (D_{B_n}, ρ_{B_n}) as shown in Figure 5.15 is an X-coloring of (D_{B_n}, ρ_{B_n}), which implies $\dim \mathrm{Col}_X(D_{B_n}, \rho_{B_n}) \geq n + 1$. By Corollary 5.3.2, we obtain $n \leq u(B_n)$. On the other hand, we can deform B_n into a trivial handlebody-knot by the crossing changes at n crossings surrounded by dotted circles depicted in Figure 5.15. Therefore it follows that $u(B_n) = n$.

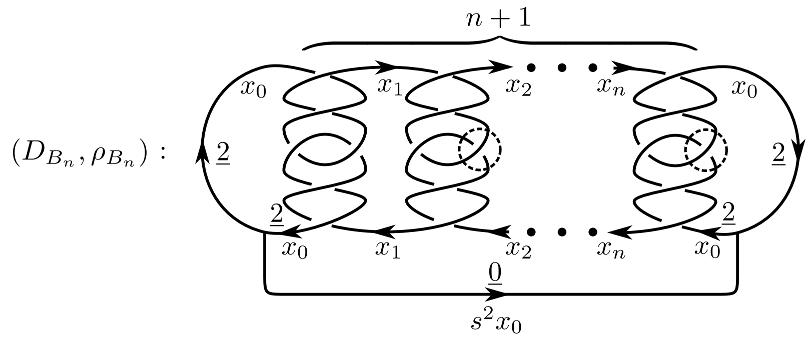


Figure 5.15: A $\mathbb{Z}_{24}$-flowed diagram (D_{B_n}, ρ_{B_n}) of B_n.

3. Let $s = 2t-1 \in \mathbb{Z}_3[t^{\pm 1}]$ and let $f(t) = t^2+t+2 \in \mathbb{Z}_3[t^{\pm 1}]$, which is an irreducible polynomial. Then $X := \mathbb{Z}_3[t^{\pm 1}]/(f(t))$ is a $\mathbb{Z}_8$-family of Alexander biquandles. Then for any $x_0, x_1, \ldots, x_n \in X$, the assignment of them to each semi-arc of (D_{C_n}, ρ_{C_n}) as shown in Figure 5.16 is an X-coloring of (D_{C_n}, ρ_{C_n}), which implies $\dim \mathrm{Col}_X(D_{C_n}, \rho_{C_n}) \geq n+1$. By Corollary 5.3.2, we obtain $n \leq u(C_n)$. On the

other hand, we can deform C_n into a trivial handlebody-knot by the crossing changes at n crossings surrounded by dotted circles depicted in Figure 5.16. Therefore it follows that $u(C_n) = n$.

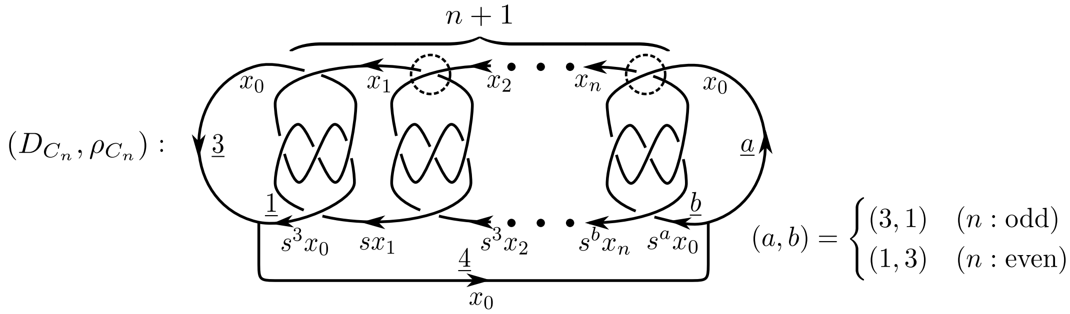


Figure 5.16: A $\mathbb{Z}_8$-flowed diagram (D_{C_n}, ρ_{C_n}) of C_n.

Example 5.4.4. Let H_n and H'_n be the handlebody-knots represented by the $\mathbb{Z}_3$-flowed diagrams (D_n, ρ_n) and $(D'_n, \rho'_n(a,b))$ respectively depicted in Figure 5.17 for any $n \in \mathbb{Z}_{>0}$ and $a, b \in \mathbb{Z}_3$. Then we show that $d(H_n, H'_n) = n$.

Let $s = 1 \in \mathbb{Z}_2[t^{\pm 1}]$ and let $f(t) = t^2 + t + 1 \in \mathbb{Z}_2[t^{\pm 1}]$, which is an irreducible polynomial. Then $X := \mathbb{Z}_2[t^{\pm 1}]/(f(t))$ is a $\mathbb{Z}_3$-family of Alexander (bi)quandles. Then for any $x_0, x_1, \dots, x_n, y_1, \dots, y_n \in X$, the assignment of them to each semi-arc of (D_n, ρ_n) as shown in Figure 5.17 is an X-coloring of (D_n, ρ_n), which implies $\dim \mathrm{Col}_X(D_n, \rho_n) \geq 2n + 1$.

On the other hand, we note that $\mathrm{Col}_X(D'_n, \rho'_n(a,b))$ is generated by $x_0, x_1, x'_1, \dots, x_n, x'_n, y_1, y'_1, \dots, y_n, y'_n \in X$ as shown in Figure 5.17 for any $a, b \in \mathbb{Z}_3$. If $(a,b) = (0,0)$, it is easy to see that $\dim \mathrm{Col}_X(D'_n, \rho'_n(a,b)) = 1$. If $(a,b) = (1,1), (1,2), (2,1), (2,2)$, we obtain that $x_i = x'_i = y_i = y'_i$ for any $i = 1, 2, \dots, n$, which implies $\dim \mathrm{Col}_X(D'_n, \rho'_n(a,b)) \leq n + 1$. If $(a,b) = (0,1), (0,2)$, we have

$$\begin{aligned}
&x_0 = x_1 = x_2, \\
&x_{i+2} = x'_i \ (i = 1, 2, \dots, n-2), \\
&x'_i = \begin{cases} x_i \underline{*}^b\, y'_i \ (i: \text{odd}), \\ x_i \underline{*}^{-b}\, y'_i \ (i: \text{even}), \end{cases} \\
&x_n = x'_{n-1}, \\
&y_i = y'_i \ (i = 1, 2, \dots, n).
\end{aligned}$$

Hence $\mathrm{Col}_X(D'_n, \rho'_n(a,b))$ is generated by $x_0, y_1, \dots, y_n \in X$, which implies $\dim \mathrm{Col}_X(D'_n, \rho'_n(a,b)) \leq n + 1$. If $(a,b) = (1,0), (2,0)$, in the same way as when $(a,b) = (0,1), (0,2)$, $\mathrm{Col}_X(D'_n, \rho'_n(a,b))$ is generated by $x_0, x_1, \dots x_n \in X$, which implies $\dim \mathrm{Col}_X(D'_n, \rho'_n(a,b)) \leq n + 1$. Hence for any $a, b \in \mathbb{Z}_3$, $\dim \mathrm{Col}_X(D'_n, \rho'_n(a,b)) \leq n + 1$, which implies that

$$\dim \mathrm{Col}_X(D_n, \rho_n) - \dim \mathrm{Col}_X(D'_n, \rho'_n(a,b)) \geq n.$$

By Theorem 5.3.1, it follows that $n \leq d(H_n, H'_n)$.

Finally, we can deform H'_n into H_n by the crossing changes at n crossings surrounded by dotted circles depicted in Figure 5.17. Therefore it follows that $d(H_n, H'_n) = n$.

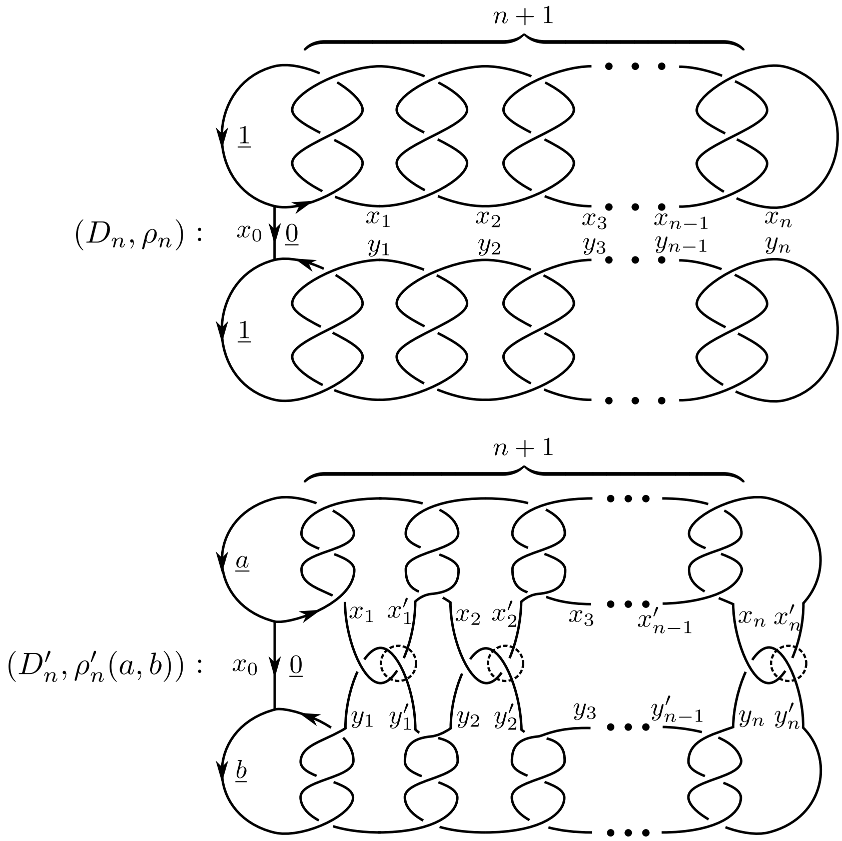


Figure 5.17: $\mathbb{Z}_3$-flowed diagrams (D_n, ρ_n) and (D'_n, ρ'_n) of H_n and H'_n.

Chapter 6

The tunnel number and the cutting number with constituent handlebody-knots

The tunnel number of a knot K in the 3-sphere S^3 is defined to be the minimal number of mutually disjoint arcs $\gamma_1, \ldots, \gamma_t$ properly embedded in $E(K)$ such that $E(K \cup \gamma_1 \cup \cdots \cup \gamma_t)$ becomes a handlebody, where $E(\cdot)$ denotes the exterior. We call the collection of the arcs $\{\gamma_1, \ldots, \gamma_t\}$ an unknotting tunnel system for K. The study of the tunnel number of knots is closely related to that of hyperbolic structures, Heegaard splittings and its Goeritz groups and so on of the exterior. Indeed, for a knot K, each unknotting tunnel system $\{\gamma_1, \ldots, \gamma_t\}$ of K provides a genus t Heegaard splitting of $E(K)$, and any genus t Heegaard splitting of $E(K)$ is obtained in this manner. In addition, many results concerning the additivity of tunnel number of knots under connected sum are often obtained through discussions on Heegaard splittings (for example, see [31, 36, 37, 38, 47, 50, etc.]). Moriah and Rubinstein [35] showed that an evaluation formula of tunnel numbers is best possible by using arguments from hyperbolic geometry. Cho and McCullough [3, 4, 5, 6] gave an effective method for the study of unknotting tunnels of knots with tunnel number 1 through discussions on Goeritz groups.

The definition of the tunnel number of knots is extended to that of handlebody-knots in the same way. The study of the handlebody-knot theory is suitable for that of unknotting tunnel systems since the operation of adding a "tunnel" has a closure property in handlebody-knot theory, that is, a handlebody-knot and its unknotting tunnel system $\{\gamma_1, \ldots, \gamma_t\}$ can be realized as a sequence of $t + 1$ handlebody-knots. Hence we can evaluate the tunnel number step by step through arguments from the handlebody-knot theory. Actually, Ishii [14] gave a lower bound for the tunnel number of handlebody-knots by using dihedral quandle colorings for handlebody-knots.

We may regard the tunnel number of a handlebody-knot H as the minimal number of 2-handles that must be "removed" from $E(H)$ such that it becomes a handlebody. In this chapter, we introduce a geometric invariant for handlebody-knots, called the cutting number, which is defined to be the minimal number of 2-handles that must be "attached" to $E(H)$ such that it becomes a handlebody. In this sense, the tunnel number and the cutting number are "dual" geometric invariants for handlebody-knots which have finite values. In this chapter, for a handlebody-knot H, we define a constituent handlebody-knot of H by a handlebody-knot obtained from H by removing an open regular neighborhood of some meridian disks of H. By introducing the notion of constituent handlebody-knots, we can deal with the tunnel number and the cutting number of handlebody-knots uniformly. In this chapter, we give necessary conditions to be constituent handlebody-knots by using G-family of quandles colorings. We also give lower bounds for the tunnel number, which is a generalization of Ishii's result in [14], and the cutting number of handlebody-knots.

The outline of the chapter is as follows. In Section 6.1, we introduce constituent handlebody-knots, the tunnel number and the cutting number of handebody-knots. In Section 6.2, we review a coloring for handlebody-knots by using a G-family of quandles and introduce a notion of trivial coloring G-flows. In Section 6.3, we consider module structures of coloring sets by G-families of Alexander quandles and give some examples of such coloring sets. In Section 6.4, we provide necessary conditions to be constituent handlebody-knots. Furthermore, as the corollaries, we give lower bounds for the tunnel number and the cutting number of handlebody-knots. In Section 6.5, we construct a family of handlebody-knots which do not contain a certain classical knot as a constituent handlebody-knot. Moreover, we construct handlebody-knots with arbitrary tunnel number and cutting number.

6.1 The tunnel number and the cutting number of handlebody-knots

In this chapter, we denote by O_g the S^1-oriented genus g trivial handlebody-knot. Let H and H' be genus g and g' ($g' < g$) handlebody-knots respectively. We call H' a *constituent handlebody-knot* of H, denoted $H' < H$, if there exists a meridian disk system $\{\Delta_1, \ldots, \Delta_g\}$ of H such that $\mathrm{cl}(H - \bigcup_{i=1}^{g-g'} N(\Delta_i)) \cong H'$, where $N(\cdot)$ and $\mathrm{cl}(\cdot)$ denote a regular neighborhood and the closure respectively. For a genus g handlebody-knot H, a subset $\{\Delta_1, \ldots, \Delta_l\}$ of a meridian disk system of H is called a *cutting system* of H if $\mathrm{cl}(H - \bigcup_{i=1}^{l} N(\Delta_i))$ is a handlebody standardly embedded in S^3, which means that the exterior is a handlebody. We note that the genus of the handlebody may be 0. Then we define the *cutting number* $\mathrm{cut}(H)$ of H by the minimal number of the cardinalities of cutting systems of H. We note that $\mathrm{cut}(O_g) = 0$ for any g. That is,

$$\mathrm{cut}(H) := \begin{cases} \min\{\#\Theta \mid \Theta : \text{a cutting system of } H\} & (H \not\cong O_g), \\ 0 & (H \cong O_g). \end{cases}$$

By the definition, the following hold.

- $\mathrm{cut}(H) = \begin{cases} g - \max\{g' \mid O_{g'} < H\} & (O_{g'} < H \text{ for some } g'), \\ g & (O_{g'} \not< H \text{ for any } g'). \end{cases}$
- $0 \leq \mathrm{cut}(H) \leq g$.
- $t(H) = \min\{i \mid H < O_{g+i}\}$,

where $t(H)$ is the tunnel number of H. The tunnel number of a handlebody-knot H, which is a well-known geometric invariant for classical knots, is defined to be the minimal number of mutually disjoint arcs $\gamma_1, \ldots, \gamma_t$ properly embedded in $E(H)$ such that $E(H \cup \gamma_1 \cup \cdots \cup \gamma_t)$ becomes a handlebody, where $E(\cdot)$ denotes the exterior. In other words, the tunnel number is the minimal number of 2-handles that must be removed from the exterior such that it becomes a handlebody. On the other hand, the cutting number of a handlebody-knot is the minimal number of 2-handles that must be attached to the exterior such that it becomes a handlebody. In this sense, we can consider the cutting number of a handlebody-knot as a dual notion to the tunnel number.

6.2 Trivial coloring G-flows

We remind a coloring for an S^1-oriented handlebody-link by a G-family of quandles. Let X be a G-family of quandles and let (D, ρ) be a G-flowed diagram of an S^1-oriented handlebody-link. An X-*coloring* of (D, ρ) is a map $C : \mathcal{A}(D, \rho) \to X$ satisfying the conditions depicted in Figure 6.1 at each crossing and vertex. An X-coloring C is *trivial* if C is a constant map. We denote by $\mathrm{Col}_X(D, \rho)$ the set of all X-colorings of (D, ρ). It is easy to see that $\#\mathrm{Col}_X(D, \rho) \geq \#X$.

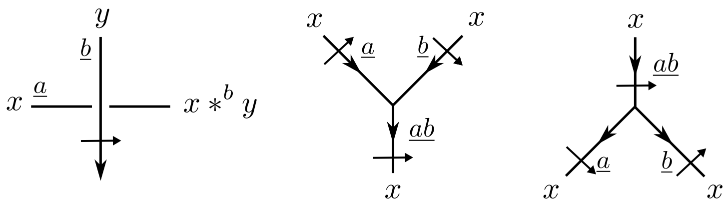


Figure 6.1: A coloring of (D, ρ) by a G-family of quandles.

Let D be a diagram of an S^1-oriented handlebody-link H. A G-flow ρ of H is a *trivial coloring G-flow* if for any G-family of quandles X and $C \in \mathrm{Col}_X(D, \rho)$, C is a trivial X-coloring. We denote by $\mathrm{Flow}_{\mathrm{trivial}}(H; G)$ the set of all trivial coloring G-flows of H.

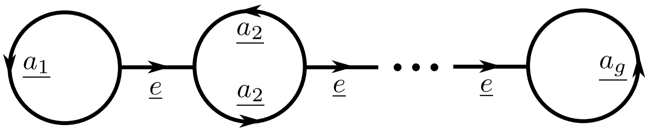
For any group G and S^1-oriented handlebody-knot H, the constant map $\rho_e : \pi_1(S^3 - H) \to G$ sending into the identity element e is a trivial coloring G-flow of H since for any G-family of quandles X and $x, y \in X$, it follows that $x *^e y = x$.

We prove the following lemma we use in Section 6.4.

Lemma 6.2.1. *For any group G, every G-flow of O_g is a trivial coloring G-flow.*

Proof. Let O_g be the diagram of the handlebody-knot O_g depicted in Figure 6.2, where we note that we use the same symbol O_g as the genus g trivial handlebody-knot. Any G-flow ρ of O_g is represented as in Figure 6.2, where $a_i \in G$ for any $i = 1, \ldots, g$ and e is the identity element of G. Hence it is easy to see that for any G-family of quandles X, every X-coloring of (O_g, ρ) is trivial. $\square$

By Lemma 6.2.1, for any G-family of quandles X and $\rho \in \mathrm{Flow}(O_g; G)$, we obtain that $\#\mathrm{Col}_X(O_g, \rho) = \#X$.



(O_g, ρ)

Figure 6.2: A G-flow of O_g.

6.3 Module structures of coloring sets by G-families of Alexander quandles

Let (D, ρ) be a G-flowed diagram of an S^1-oriented handlebody-link and let X be a G-family of Alexander quandles as a right $R[G]$-module for some ring R. Then $\mathrm{Col}_X(D, \rho)$ is a right R-module with the action $(C \cdot r)(x) = C(x)r$ and the addition $(C + C')(x) = C(x) + C'(x)$ for any $C, C' \in \mathrm{Col}_X(D, \rho)$, $x \in \mathcal{A}(D, \rho)$ and $r \in R$. In this section, we consider the module structures of coloring sets by G-families of Alexander quandles.

Let R and R' be rings. We denote by $M(m, n; R)$ the set of $m \times n$ matrices over R and set $M(n; R) := M(n, n; R)$. We denote by $GL(n; R)$ the set of $n \times n$ invertible matrices over R. We can regard a matrix in $M(m, n; M(k, l; R))$ as a matrix in $M(km, ln; R)$. We call it a *flat matrix*. For any $(a_{i,j}) \in M(m, n; R)$ and map $f : R \to R'$, we define $f((a_{i,j})) = (f(a_{i,j})) \in M(m, n; R')$.

Let R be a commutative ring, G be a group and let X be a right $R[G]$-module. Then X is also an R-module. We assume that X is a finitely generated free R-module, that is, X is isomorphic to R^d for some $d \in \mathbb{Z}_{\geq 0}$. Let $A = (a_{i,j}) \in M(n, m; R[G])$ and let $f_A : X^n \to X^m$ be an R-homomorphism defined by $f_A((x_1, \ldots, x_n)) = (x_1, \ldots, x_n)A$, where $(x_1, \ldots, x_n)A$ means $(\sum_{i=1}^n x_i a_{i,1}, \ldots, \sum_{i=1}^n x_i a_{i,m})$.

An action of G on X is a group homomorphism $\eta : G \to \mathrm{Aut}_{R\text{-}\mathsf{Mod}}(X) \cong GL(d; R)$, where R-Mod is the category of R-modules, and $\mathrm{Aut}_{R\text{-}\mathsf{Mod}}(X)$ is the automorphism group of X. Then η induces $\widetilde{\eta} : R[G] \to M(d; R)$ satisfying the commutative diagram

$$\begin{tikzcd} G \arrow[r, "\eta"] \arrow[d, hook, "\text{inclusion}"'] & \mathrm{Aut}_{R\text{-}\mathsf{Mod}}(X) \cong GL(d; R) \arrow[r, hook, "\text{inclusion}"] & M(d; R) \\ R[G] \arrow[urr, "\widetilde{\eta}"'] & & \end{tikzcd}.$$

That is, for any $(r_1, \ldots, r_d) \in R^d$ and $\sum_{g \in G} r_g g \in R[G]$,

$$(r_1, \ldots, r_d) \cdot \sum_{g \in G} r_g g = (r_1, \ldots, r_d) \sum_{g \in G} r_g \eta(g) = (r_1, \ldots, r_d) \widetilde{\eta}(\sum_{g \in G} r_g g).$$

Then it follows that

$$\begin{aligned}\operatorname{Ker} f_A &= \{(x_1,\dots,x_n)\in X^n | (x_1,\dots,x_n)A = \mathbf{0}\}\\ &\cong \Big\{((r_{1,1},\dots,r_{1,d}),\dots,(r_{n,1},\dots,r_{n,d}))\in (R^d)^n \Big| \\ &\qquad ((r_{1,1},\dots,r_{1,d}),\dots,(r_{n,1},\dots,r_{n,d}))\widetilde{\eta}(A) = \mathbf{0}\}\\ &\cong \Big\{(r_{1,1},\dots,r_{n,d})\in R^{dn} \Big| (r_{1,1},\dots,r_{n,d})\widetilde{\eta}(A) = \mathbf{0}\Big\},\end{aligned}$$

where $\widetilde{\eta}(A) \in M(n,m;M(d;R))$, and we regard $\widetilde{\eta}(A)$ as the flat matrix in $M(dn,dm;R)$ in the last line. Therefore when R is a field F, it follows that $\operatorname{Ker} f_A$ is a vector subspace of X^n over F, and $\dim_F \operatorname{Ker} f_A = dn - \operatorname{rank}\widetilde{\eta}(A)$. In particular, if X is an extension field of F, the map f_A is also an X-linear map, and $\operatorname{Ker} f_A$ is a vector subspace of X^n over X. An action of G on X is a group homomorphism $\zeta : G \to \operatorname{Aut}_{X\text{-}\mathsf{Vect}}(X) \cong X$, where X-Vect is the category of vector spaces over X. Then ζ induces $\widetilde{\zeta} : F[G] \to X$ satisfying the commutative diagram

$$\begin{tikzcd} G \arrow[r, "\zeta"] \arrow[d, hook, "\text{inclusion}"'] & \operatorname{Aut}_{X\text{-}\mathsf{Vect}}(X) \cong X . \\ F[G] \arrow[ur, "\widetilde{\zeta}"'] & \end{tikzcd}$$

That is, for any $x \in X$ and $\sum_{g\in G} k_g g \in F[G]$,

$$x \cdot \sum_{g\in G} k_g g = x \sum_{g\in G} k_g \zeta(g) = x\widetilde{\zeta}(\sum_{g\in G} k_g g).$$

Then it follows that

$$\begin{aligned}\operatorname{Ker} f_A &= \{(x_1,\dots,x_n)\in X^n | (x_1,\dots,x_n)A = \mathbf{0}\}\\ &\cong \Big\{(x_1,\dots,x_n)\in X^n \Big| (x_1,\dots,x_n)\widetilde{\zeta}(A) = \mathbf{0}\Big\},\end{aligned}$$

where $\widetilde{\zeta}(A) \in M(n,m;X)$. Therefore it follows that $\dim_X \operatorname{Ker} f_A = n - \operatorname{rank}\widetilde{\zeta}(A)$ and $d\cdot \dim_X \operatorname{Ker} f_A = \dim_F \operatorname{Ker} f_A$.

In this chapter, we assume that every component of a diagram of any S^1-oriented handlebody-link has a crossing at least 1. Let (D,ρ) be a G-flowed diagram of an S^1-oriented handlebody-link and let X be a G-family of Alexander quandles as a right $R[G]$-module for some ring R. We put $C(D,\rho) = \{c_1,\dots,c_{n_1}\}$ and $V(D,\rho) = \{\tau_1,\dots,\tau_{2n_2}\}$, where $C(D,\rho)$ and $V(D,\rho)$ are the set of all crossings of (D,ρ) and the one of all vertices of (D,ρ) respectively, and the sign of τ_i is 1 for any $i = 1,\dots,n_2$ and -1 for any $i = n_2+1,\dots,2n_2$. Put $n := n_1 + 3n_2$. We denote by x_i each arc of (D,ρ) as shown in Figure 6.3, which implies $\mathcal{A}(D,\rho) = \{x_1,\dots,x_n\}$. We denote by $u_i, v_i, w_i, \alpha_i, \beta_i$ and γ_i the arcs incident to a crossing c_i or a vertex τ_i as shown in Figure 6.4.

For any arcs $x, x' \in \mathcal{A}(D,\rho)$, we put

$$\delta(x,x') := \begin{cases} 1 & (x = x'),\\ 0 & (x \neq x').\end{cases}$$

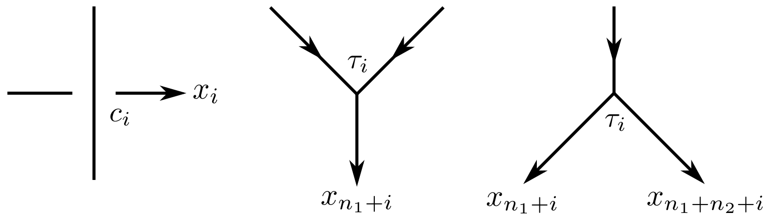


Figure 6.3: Arcs.

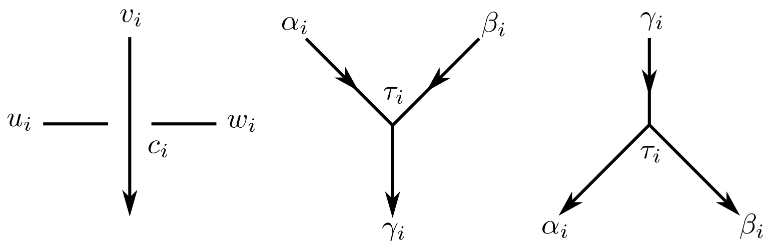


Figure 6.4: Notations.

Then we define a matrix $A(D,\rho;X) = (a_{i,j}) \in M(n_1+4n_2, n; R[G])$ by

$$a_{i,j} = \begin{cases} \delta(u_i,x_j)\rho(v_i) + \delta(v_i,x_j)(e-\rho(v_i)) - \delta(w_i,x_j) & (1 \leq i \leq n_1), \\ \delta(\alpha_{i-n_1},x_j) - \delta(\gamma_{i-n_1},x_j) & (n_1+1 \leq i \leq n_1+2n_2), \\ \delta(\beta_{i-n_1-2n_2},x_j) - \delta(\gamma_{i-n_1-2n_2},x_j) & (n_1+2n_2+1 \leq i \leq n_1+4n_2). \end{cases}$$

We note that $A(D,\rho;X)$ is determined up to permuting of rows and columns of the matrix. Then we can identify $\mathrm{Col}_X(D,\rho)$ with the right R-module

$$\left\{(z_1,\ldots,z_n) \in X^n \middle| (z_1,\ldots,z_n)A(D,\rho;X)^T = \mathbf{0}\right\}$$

with the action $(z_1,\ldots,z_n)r = (z_1r,\ldots,z_nr)$ for any $(z_1,\ldots,z_n) \in \mathrm{Col}_X(D,\rho)$ and $r \in R$, where $A(D,\rho;X)^T$ is the transposed matrix of $A(D,\rho;X)$. Hence if R is a commutative ring and $X \cong R^d$ as R-modules for some $d \in \mathbb{Z}_{\geq 0}$, it follows that $\mathrm{Col}_X(D,\rho) \cong \mathrm{Ker}\, f_{A(D,\rho;X)^T}$, where we remind that $f_{A(D,\rho;X)^T} : X^n \to X^{n_1+4n_2}$ is an R-homomorphism defined by $f_{A(D,\rho;X)^T}(z_1,\ldots,z_n) = (z_1,\ldots,z_n)A(D,\rho;X)^T$.

For example, let (E,ψ) be the G-flowed diagram of the handlebody-knot depicted in Figure 6.5. Then for a G-family of Alexander quandles X as a right $R[G]$-module, we have

$$A(E,\psi;X) = \begin{pmatrix} b & 0 & e-b & 0 & -1 \\ e-a & a & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 \end{pmatrix} \in M(6,5;R[G])$$

and

$$\mathrm{Col}_X(E,\psi) \cong \left\{(z_1,\ldots,z_5)\in X^5 \middle| (z_1,\ldots,z_5)A(E,\psi;X)^T = \mathbf{0}\right\}.$$

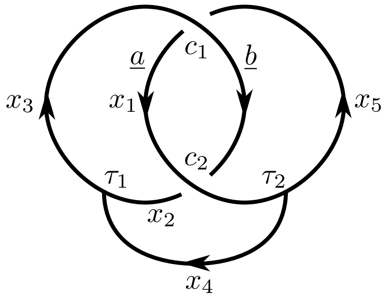


Figure 6.5: A G-flowed diagram (E,ψ).

Example 6.3.1. Let X be an Alexander quandle as an $R[t^{\pm 1}]$-module for some commutative ring R and put $k := \mathrm{type}\, X$. Then X is an $R[\mathbb{Z}_k]$-module with $x\cdot t^i = xt^i$ for any $x\in X$ and $t^i \in \mathbb{Z}_k$, where we regard $\mathbb{Z}_k$ as $\langle t \mid t^k\rangle$. Hence X is a $\mathbb{Z}_k$-family of Alexander quandles. Therefore for a $\mathbb{Z}_k$-flowed diagram (D,ρ) of an S^1-oriented handlebody-link, $\mathrm{Col}_X(D,\rho)$ is an R-module. When R is a field F and $X\cong F^d$ as vector spaces over F for some $d\in\mathbb{Z}_{\geq 0}$, it follows that $\mathrm{Col}_X(D,\rho)$ is a vector space over F, and $\dim_F \mathrm{Col}_X(D,\rho) = dn - \mathrm{rank}\,\widetilde{\eta}(A(D,\rho;X))$, where $n = \#\mathcal{A}(D,\rho)$. In particular, if X is an extension field of F, it follows that $\mathrm{Col}_X(D,\rho)$ is also a vector space over X, and $\dim_X \mathrm{Col}_X(D,\rho) = n - \mathrm{rank}\,\widetilde{\zeta}(A(D,\rho;X))$.

Example 6.3.2. Let R be a ring, $X = R^d$ and $G = GL(d;R)$ for some $d\in\mathbb{Z}_{\geq 0}$. Then X is a right $R[G]$-module with $(r_1,\ldots,r_d)\cdot(a_{i,j}) = (\sum_{i=1}^d r_i a_{i,1},\ldots,\sum_{i=1}^d r_i a_{i,d})$ for any $(r_1,\ldots,r_d)\in X$ and $(a_{i,j})\in G$. Hence X is a G-family of Alexander quandles. Therefore for a G-flowed diagram (D,ρ) of an S^1-oriented handlebody-link, $\mathrm{Col}_X(D,\rho)$ is a right R-module. When R is a field F, it follows that $\mathrm{Col}_X(D,\rho)$ is a vector space over F, and $\dim_F \mathrm{Col}_X(D,\rho) = dn - \mathrm{rank}\,\widetilde{\eta}(A(D,\rho;X))$, where $n = \#\mathcal{A}(D,\rho)$.

6.4 Results

In this section, we provide essential conditions to be constituent handlebody-knots by using colorings by G-families of quandles. Furthermore, as the corollaries, we give lower bounds for the tunnel number and the cutting number of handlebody-knots.

Theorem 6.4.1. *Let H and H' be S^1-oriented genus g and g' $(g'<g)$ handlebody-knots and D and D' be their diagrams respectively. Let $\rho'\in\mathrm{Flow}(H',G)$ and X be a G-family of Alexander quandles as a right $F[G]$-module for some field F, where $X\cong$*

F^d *as vector spaces over* F *for some* $d \in \mathbb{Z}_{\geq 0}$. *If* $H' < H$, *there exists* $\rho \in \mathrm{Flow}(H; G)$ *such that* $\mathrm{Im}\,\rho = \mathrm{Im}\,\rho'$ *and*

$$\dim_F \mathrm{Col}_X(D', \rho') - \dim_F \mathrm{Col}_X(D, \rho) \leq d(g - g').$$

Proof. Assume that $H' < H$ and put $m := g - g'$. There exist S^1-oriented handlebody-knots $H_0, H_1, \ldots, H_m$ such that $H_0 = H'$, $H_m = H$, $H_i < H_{i+1}$ for any $i = 0, 1, \ldots, m-1$, and the genus of H_i is $g' + i$ for any $i = 0, 1, \ldots, m$. For any $\rho_i \in \mathrm{Flow}(H_i; G)$, the handlebody-knots H_i and H_{i+1} respectively have G-flowed diagrams (D_i, ρ_i) and (D_{i+1}, ρ_{i+1}) which are identical except in the neighborhood of a point where they differ as shown in Figure 6.6. Here we may assume that the two arcs of (D_i, ρ_i) in the left of Figure 6.6 are x_1 and x_2 ($x_1 \neq x_2$), where we put $\mathcal{A}(D_i, \rho_i) = \{x_1, \ldots, x_n\}$. It is easy to see that $\mathrm{Im}\,\rho_i = \mathrm{Im}\,\rho_{i+1}$. Then we have

$$\mathrm{Col}_X(D_i, \rho_i) \cong \left\{(z_1, \ldots, z_n) \in X^n \middle| (z_1, \ldots, z_n) A(D_i, \rho_i; X)^T = \mathbf{0}\right\}$$

as vector spaces over F. Since the coloring set $\mathrm{Col}_X(D_{i+1}, \rho_{i+1})$ is obtained from $\mathrm{Col}_X(D_i, \rho_i)$ by adding one relation $z_1 = z_2$, we have

$$\begin{aligned}\mathrm{Col}_X(D_{i+1}, \rho_{i+1}) &\cong \left\{(z_1, \ldots, z_n) \in X^n \middle| (z_1, \ldots, z_n) A(D_i, \rho_i; X)^T = \mathbf{0}, z_1 = z_2\right\} \\ &\cong \left\{(z_1, \ldots, z_n) \in X^n \middle| (z_1, \ldots, z_n) \begin{pmatrix} A(D_i, \rho_i; X) \\ \boldsymbol{a} \end{pmatrix}^T = \mathbf{0}\right\}\end{aligned}$$

as vector spaces over F, where $\boldsymbol{a} = (e, -e, 0, \ldots, 0)$. We note that $\widetilde{\eta}(e)$ is the $d \times d$ identity matrix. Therefore it follows that

$$0 \leq \mathrm{rank}\,\widetilde{\eta}(\begin{pmatrix} A(D_i, \rho_i; X) \\ \boldsymbol{a} \end{pmatrix}^T) - \mathrm{rank}\,\widetilde{\eta}(A(D_i, \rho_i; X)^T) \leq d$$

as flat matrices. Therefore we obtain that

$$0 \leq \dim_F \mathrm{Col}_X(D_i, \rho_i) - \dim_F \mathrm{Col}_X(D_{i+1}, \rho_{i+1}) \leq d.$$

Consequently, for any $\rho' \in \mathrm{Flow}(H', G)$, there exists $\rho \in \mathrm{Flow}(H; G)$ such that $\mathrm{Im}\,\rho = \mathrm{Im}\,\rho'$ and $\dim_F \mathrm{Col}_X(D', \rho') - \dim_F \mathrm{Col}_X(D, \rho) \leq dm = d(g - g')$. $\square$

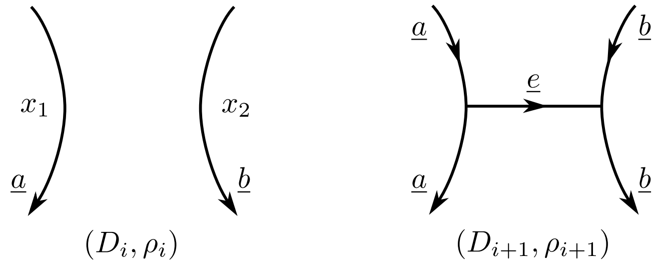


Figure 6.6: Adding an arc.

Theorem 6.4.2. *Let H and H' be S^1-oriented genus g and g' $(g' < g)$ handlebody-knots respectively and let G be a group. If $H' < H$, it follows that*

$$\#\mathrm{Flow}_{\mathrm{trivial}}(H'; G) \leq \#\mathrm{Flow}_{\mathrm{trivial}}(H; G).$$

Proof. Assume that $H' < H$ and put $m := g - g'$. There exist S^1-oriented handlebody-knots $H_0, H_1, \ldots, H_m$ such that $H_0 = H'$, $H_m = H$, $H_i < H_{i+1}$ for any $i = 0, 1, \ldots, m-1$, and the genus of H_i is $g' + i$ for any $i = 0, 1, \ldots, m$. For any $\rho_i \in \mathrm{Flow}_{\mathrm{trivial}}(H_i; G)$, the handlebody-knots H_i and H_{i+1} respectively have a trivial coloring G-flowed diagram (D_i, ρ_i) and a G-flowed diagram (D_{i+1}, ρ_{i+1}) which are identical except in the neighborhood of a point where they differ as shown in Figure 6.6. Assume that ρ_{i+1} is not a trivial coloring G-flow of H_{i+1}, which means that there exists a G-family of quandles X and a non-trivial X-coloring C of (D_{i+1}, ρ_{i+1}). Then C induces a non-trivial X-coloring of (D_i, ρ_i), that is, the X-coloring of (D_i, ρ_i) obtained from C by ignoring the arc we added as shown in Figure 6.6 is not trivial. This contradicts to $\rho_i \in \mathrm{Flow}_{\mathrm{trivial}}(H_i; G)$. Hence ρ_{i+1} is a trivial coloring G-flow of H_{i+1}, which implies that $\#\mathrm{Flow}_{\mathrm{trivial}}(H_i; G) \leq \#\mathrm{Flow}_{\mathrm{trivial}}(H_{i+1}; G)$. Consequently, we obtain that $\#\mathrm{Flow}_{\mathrm{trivial}}(H'; G) \leq \#\mathrm{Flow}_{\mathrm{trivial}}(H; G)$. □

By Theorems 6.4.1 and 6.4.2, we have the following corollaries concerning evaluations of the tunnel number and the cutting number of handlebody-knots.

Corollary 6.4.3. *Let H be an S^1-oriented handlebody-knot and (D, ρ) be a G-flowed diagram of H. Let X be a G-family of Alexander quandles as a right $F[G]$-module for some field F, where $X \cong F^d$ as vector spaces over F for some $d \in \mathbb{Z}_{\geq 0}$. Then it follows that*

$$\frac{\dim_F \mathrm{Col}_X(D, \rho)}{d} - 1 \leq t(H).$$

Proof. Put $m := t(H)$, which implies that $H < O_{g+m}$. Let (O_g, ρ_0) be a G-flowed diagram of O_g, where we note that we use the same symbol O_g as the genus g trivial handlebody-knot. By Lemma 6.2.1, we have $\dim_F \mathrm{Col}_X(O_g, \rho_o) = d$. By Theorem 6.4.1, we obtain that $\dim_F \mathrm{Col}_X(D, \rho) - d \leq dm$, which completes the proof. □

Corollary 6.4.4. *Let H be an S^1-oriented genus g handlebody-knot and let G be a group. Then it follows that*

$$g - \log_{|G|} \#\mathrm{Flow}_{\mathrm{trivial}}(H; G) \leq \mathrm{cut}(H).$$

Proof. Put $m := \mathrm{cut}(H)$ and suppose $m < g$. Then we have $O_{g-m} < H$. By Lemma 6.2.1, we have $\mathrm{Flow}_{\mathrm{trivial}}(O_{g-m}; G) = \mathrm{Flow}(O_{g-m}; G) = |G|^{g-m}$. Therefore, by Theorem 6.4.2, it follows that $|G|^{g-m} \leq \#\mathrm{Flow}_{\mathrm{trivial}}(H; G)$, which implies that $g - \log_{|G|} \#\mathrm{Flow}_{\mathrm{trivial}}(H; G) \leq m$. When $m = g$, we immediately obtain that $g - \log_{|G|} \#\mathrm{Flow}_{\mathrm{trivial}}(H; G) \leq m$. This completes the proof. □

6.5 Examples

In this section, we give some examples. In Example 6.5.1, we construct a family of handlebody-knots which do not contain a certain knot as a constituent handlebody-knot. In Example 6.5.2, we give a family of genus g handlebody-knots with tunnel number gn for any $g \in \mathbb{Z}_{>0}$ and $n \in \mathbb{Z}_{\geq 0}$. In Example 6.5.3, we give a family of genus g handlebody-knots with cutting number g for any $g \in \mathbb{Z}_{\geq 2}$.

Example 6.5.1. Let K and H_n be respectively the knot and the genus 2 handlebody-knot represented by the $\mathbb{Z}_2$-flowed diagrams (D, ρ) and $(D_n, \rho_n(a, b))$ depicted in Figure 6.7 for any $n \in \mathbb{Z}_{\geq 0}$ and $a, b \in \mathbb{Z}_2$. We note that K is the knot 8_{18} in Rolfsen's knot table [46], and H_1 is the genus 2 handlebody-knot 5_4 in the table given in [21]. Let X be the Alexander quandle $\mathbb{Z}_3[t^{\pm 1}]/(t + 1)$, which is isomorphic to the field $\mathbb{Z}_3$. Since $\operatorname{type} X = 2$, X is the $\mathbb{Z}_2$-family of Alexander quandles. Then for any $z_1, z_2, z_3 \in X$, the assignment of them to each arc of (D, ρ) as shown in Figure 6.7 is an X-coloring of (D, ρ), which implies that $\dim_{\mathbb{Z}_3} \operatorname{Col}_X(D, \rho) \geq 3$. On the other hand, we can easily see that for any $n \in \mathbb{Z}_{\geq 0}$ and $a, b \in \mathbb{Z}_2$, each X-coloring of $(D_n, \rho_n(a, b))$ is trivial, which implies that $\dim_{\mathbb{Z}_3} \operatorname{Col}_X(D_n, \rho_n(a, b)) = 1$. Hence we have $\dim_{\mathbb{Z}_3} \operatorname{Col}_X(D, \rho) - \dim_{\mathbb{Z}_3} \operatorname{Col}_X(D_n, \rho_n(a, b)) \geq 2$ for any $n \in \mathbb{Z}_{\geq 0}$ and $a, b \in \mathbb{Z}_2$. Therefore we obtain that $K \not< H_n$ for any $n \in \mathbb{Z}_{\geq 0}$ by Theorem 6.4.1.

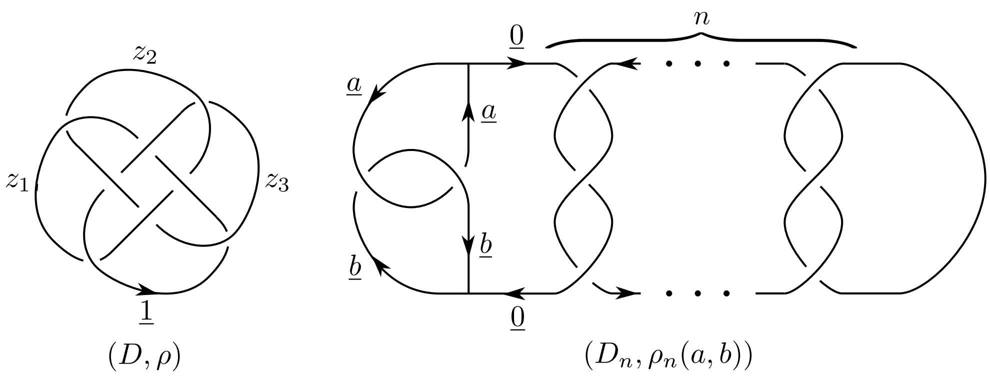


Figure 6.7: $\mathbb{Z}_2$-flowed diagrams (D, ρ) and $(D_n, \rho_n(a, b))$ of K and H_n respectively.

Example 6.5.2. Let $H_{g,n}$ be the S^1-oriented genus g handlebody-knot represented by the $\mathbb{Z}_3$-flowed diagram $(D_{g,n}, \rho_{g,n})$ depicted in Figure 6.8 for any $g \in \mathbb{Z}_{>0}$ and $n \in \mathbb{Z}_{\geq 0}$. Let X be the Alexander quandle $\mathbb{Z}_2[t^{\pm 1}]/(t^2 + t + 1)$, which is an extension field of $\mathbb{Z}_2$ and isomorphic to $(\mathbb{Z}_2)^2$ as vector spaces over $\mathbb{Z}_2$. Since $\operatorname{type} X = 3$, X is the $\mathbb{Z}_3$-family of Alexander quandles. Then for any $z_0, z_{i,j} \in X$ $(1 \leq i \leq g, 1 \leq j \leq n)$, the assignment of them to each arc of $(D_{g,n}, \rho_{g,n})$ as shown in Figure 6.8 is an X-coloring of $(D_{g,n}, \rho_{g,n})$, which implies that

$$\dim_{\mathbb{Z}_2} \operatorname{Col}_X(D_{g,n}, \rho_{g,n}) = 2 \dim_X \operatorname{Col}_X(D_{g,n}, \rho_{g,n}) \geq 2(gn + 1).$$

Hence it follows that $gn \leq t(H_{g,n})$ by Corollary 6.4.3. On the other hand, the set of gn arcs drawn by a dotted line in Figure 6.8 is an unknotting tunnel system for $H_{g,n}$. Therefore we obtain that $t(H_{g,n}) = gn$ for any $g \in \mathbb{Z}_{>0}$ and $n \in \mathbb{Z}_{\geq 0}$.

Example 6.5.3. For any $g \geq 2$ and $l_1, \ldots, l_g \in 2\mathbb{Z}$, let $H_{l_1,\ldots,l_g}$ be the genus g handlebody-knot represented by the spatial graph $\Gamma_{l_1,\ldots,l_g}$, which is a graph embedded in S^3, with a g-valent vertex v_g depicted in Figure 6.9, which means that

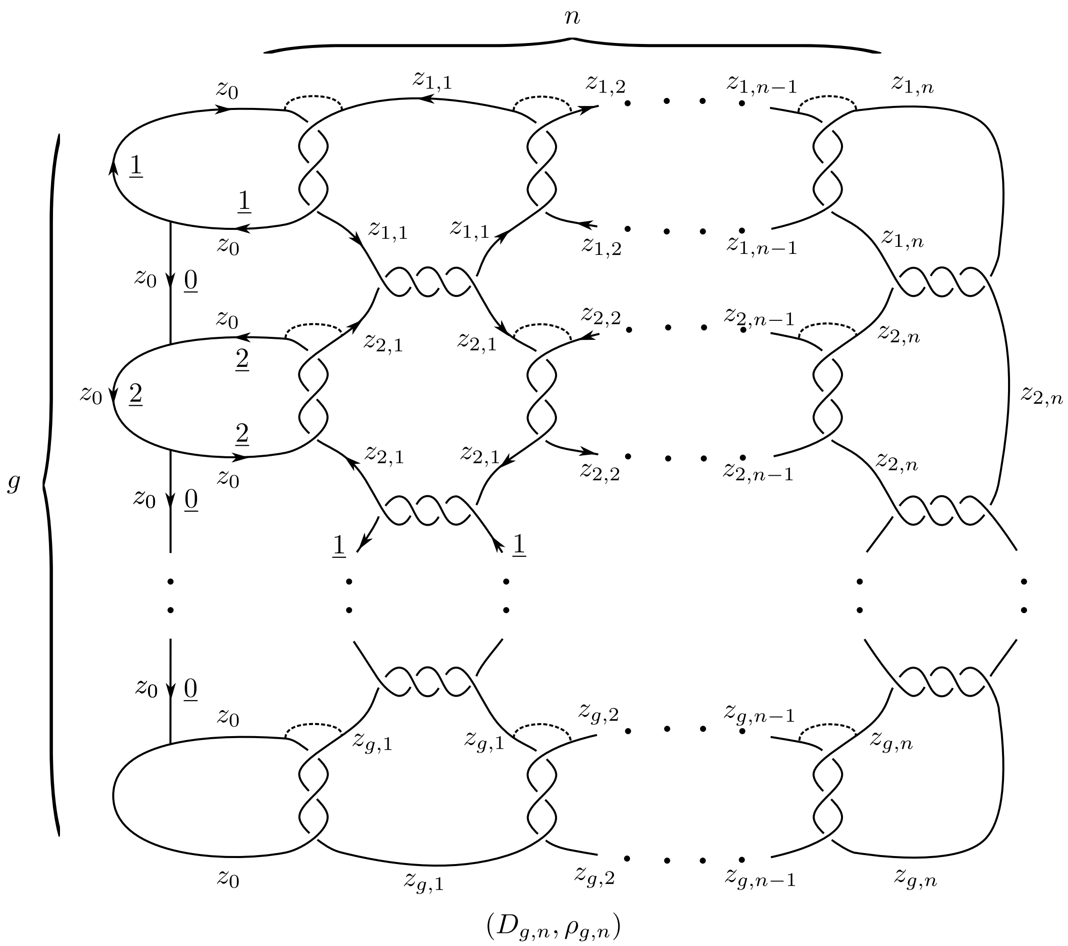


Figure 6.8: A $\mathbb{Z}_3$-flowed diagram $(D_{g,n}, \rho_{g,n})$ of $H_{g,n}$.

$H_{l_1,\dots,l_g}$ is a regular neighborhood of $\Gamma_{l_1,\dots,l_g}$. $H_{l_1,\dots,l_g}$ has the $\mathbb{Z}_2$-flowed diagram $(D_{l_1,\dots,l_g}, \rho(a_1,\dots,a_g))$ depicted in Figure 6.10 for any $a_i \in \mathbb{Z}_2$. We note that $\mathrm{Flow}(H_{l_1,\dots,l_g};\mathbb{Z}_2) = \{\rho(a_1,\dots,a_g) \mid a_i \in \mathbb{Z}_2\}$. Let X be the Alexander quandle $\mathbb{Z}_3[t^{\pm 1}]/(t+1)$, which is isomorphic to the field $\mathbb{Z}_3$. Since $\mathrm{type}\,X = 2$, X is the $\mathbb{Z}_2$-family of Alexander quandles. Suppose that $(a_1,\dots,a_g) \neq (0,\dots,0)$. Since $\Gamma_{l_1,\dots,l_g}$ has a g-fold rotational symmetry to v_g, we may assume that $a_1 = 1$. First, if $l_1 = 4l$ for some $l \in \mathbb{Z}$, we have the non-trivial X-coloring of $(D_{l_1,\dots,l_g}, \rho(a_1,\dots,a_g))$ depicted in the top of Figure 6.11. Next, if $l_1 = 4l+2$ for some $l \in \mathbb{Z}$ and $a_2 = 0$, we have the non-trivial X-coloring of $(D_{l_1,\dots,l_g}, \rho(a_1,\dots,a_g))$ depicted in the middle of Figure 6.11. Finally, if $l_1 = 4l+2$ for some $l \in \mathbb{Z}$ and $a_2 = 1$, we have the non-trivial X-coloring of $(D_{l_1,\dots,l_g}, \rho(a_1,\dots,a_g))$ depicted in the bottom of Figure 6.11. Hence we have $\mathrm{Flow}_{\mathrm{trivial}}(H_{l_1,\dots,l_g};\mathbb{Z}_2) = \{\rho(0,\dots,0)\}$, that is, $\#\mathrm{Flow}_{\mathrm{trivial}}(H_{l_1,\dots,l_g};\mathbb{Z}_2) = 1$. Therefore, we obtain that $g \leq \mathrm{cut}(H_{l_1,\dots,l_g})$, which implies that $\mathrm{cut}(H_{l_1,\dots,l_g}) = g$ for any $g \geq 2$ and $l_1,\dots,l_g \in 2\mathbb{Z}$ by Corollary 6.4.4.

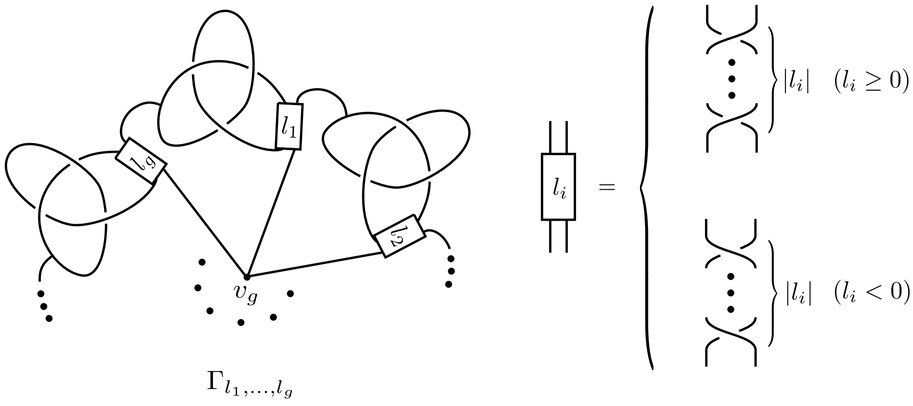


Figure 6.9: A spatial graph $\Gamma_{l_1,\ldots,l_g}$.

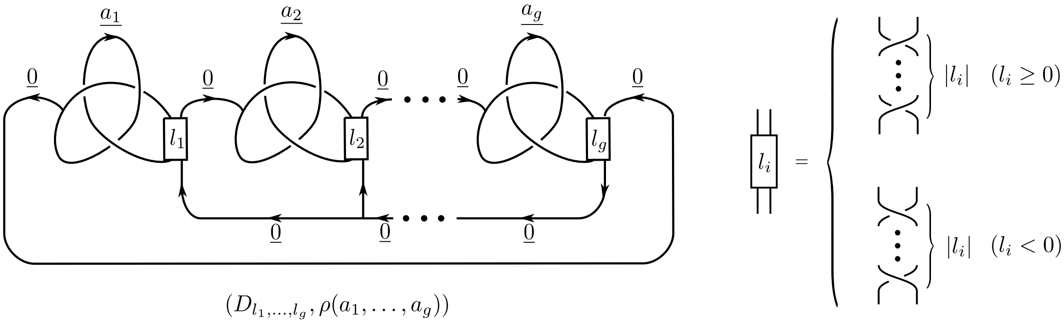
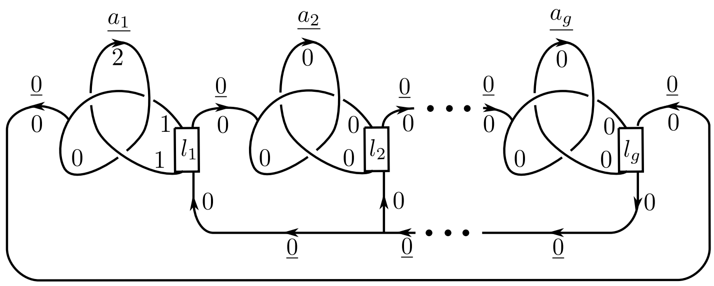


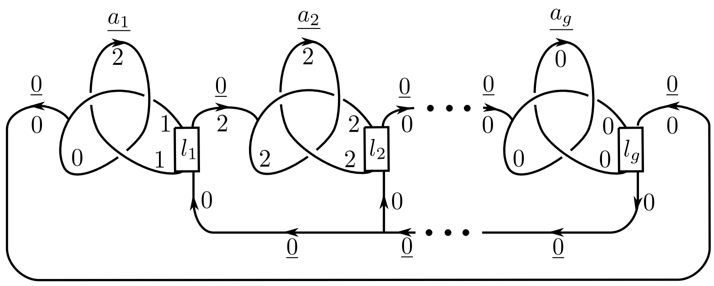
Figure 6.10: A $\mathbb{Z}_2$-flowed diagram $(D_{l_1,\ldots,l_g}, \rho(a_1,\ldots,a_g))$ of $H_{l_1,\ldots,l_g}$.

Case I : $l_1 = 4l$ $(l \in \mathbb{Z})$,
$a_1 = 1$



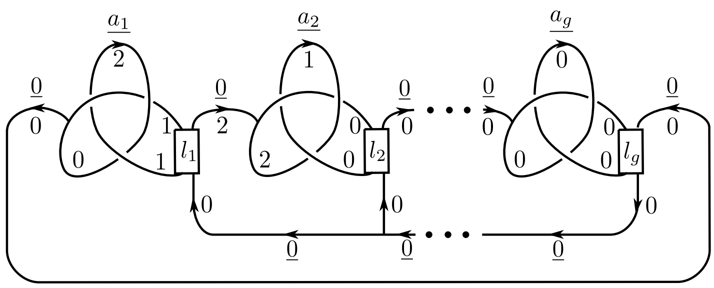
$(D_{l_1,\ldots,l_g}, \rho(a_1, \ldots, a_g))$

Case II : $l_1 = 4l + 2$ $(l \in \mathbb{Z})$,
$a_1 = 1$, $a_2 = 0$



$(D_{l_1,\ldots,l_g}, \rho(a_1, \ldots, a_g))$

Case III : $l_1 = 4l + 2$ $(l \in \mathbb{Z})$,
$a_1 = 1$, $a_2 = 1$



$(D_{l_1,\ldots,l_g}, \rho(a_1, \ldots, a_g))$

Figure 6.11: Non-trivial X-colorings of $(D_{l_1,\ldots,l_g}, \rho(a_1, \ldots, a_g))$.

Chapter 7

A relationship between multiple conjugation quandle/biquandle colorings

A quandle [27, 34] is an algebraic system whose axioms are derived from the Reidemeister moves on oriented link diagrams, and a biquandle [9, 10, 30] is a generalization of a quandle. The two algebraic systems yield many invariants for not only classical links but also surface links, virtual links and so on. In particular, some invariants obtained from biquandles are stronger than those obtained from quandles for virtual links [29]. On the other hand, as a corollary of [48], it follows that there is a one-to-one correspondence between the set of biquandle colorings and that of quandle colorings for any classical links, where in the proof of the statement, any classical link need be represented by a closed braid diagram.

Recently, Ishikawa [24] constructed a left adjoint functor $\mathcal{B}$ of a functor $\mathcal{Q}$ from the category of biquandles to that of quandles which is defined in [2]. By using $\mathcal{B}$, he proved that we can reconstruct a fundamental biquandle from a fundamental quandle, and there is a one-to-one correspondence between the set of biquandle colorings and that of quandle colorings for any classical and surface links, where in the statement, we can choose any diagram for classical and surface links. Here we note that any left adjoint functor of the functor $\mathcal{Q}$ from the category of MCBs to that of MCQs, which we will define in Section 7.2, has not been defined yet. Furthermore, Ishikawa and Tanaka [25] explained the one-to-one correspondence proved in [24] diagrammatically and concretely for classical and surface links. On the other hand, for handlebody-links, although MCB colorings require more calculation than MCQ colorings in general, it has not been known whether an invariant obtained from MCB colorings is more effective than one obtained from MCQ colorings.

In this chapter, we partially extend the result in [24, 25] to MCQ and MCB colorings for handlebody-links and spatial trivalent graphs. Concretely, we show that for any handlebody-links and spatial trivalent graphs, there is a one-to-one correspondence between the set of MCB colorings and that of MCQ colorings diagrammatically. We also show that the set of G-family of Aleander biquandles colorings is isomorphic to that of G-family of Alexander quandles colorings as modules.

This chapter is organized into three sections. In Section 7.1, we recall basic notions and facts about quandles and biquandles. In Section 7.2, we define a functor $\mathcal{Q}$ from the category of MCBs to that of MCQs and show that for any MCB X, there is a one-to-one correspondence between the set of X-colorings and that of $\mathcal{Q}(X)$-colorings diagrammatically for any handlebody-link and spatial trivalent graph. In Section 7.3, we discuss the similar correspondence between the sets of colorings by using a G-family of quandles and a G-family of biquandles.

7.1 A relationship between quandle/biquandle colorings

For any biquandle $(X, \underline{*}, \overline{*})$, we have a quandle $(X, *)$, denoted by $\mathcal{Q}(X)$, by defining $x * y = x \,\underline{*}\, y \,\overline{*}^{-1}\, y$ for any $x, y \in X$ [2]. This gives a one-to-one correspondence between the set of X-colorings and that of $\mathcal{Q}(X)$-colorings for any classical link [48] and surface link [24, 25].

For any Alexander biquandle X, which is an $R[s^{\pm 1}, t^{\pm 1}]$-module for some commutative ring R, $\mathcal{Q}(X)$ is the Alexander quandle, which is the $R[(s^{-1}t)^{\pm 1}]$-module. That is, for any $x, y \in \mathcal{Q}(X)$, it follows that $x * y = s^{-1}tx + (1 - s^{-1}t)y$.

Proposition 7.1.1. *For any Alexander biquandle X which is of finite type,* $\operatorname{type} X$ *is divisible by* $\operatorname{type} \mathcal{Q}(X)$.

Proof. Put $m = \operatorname{type} X$ and $m' = \operatorname{type} \mathcal{Q}(X)$. Then it follows that $x \,\underline{*}^{[m]}\, y = t^m x + (s^m - t^m)y = x$ and $x \,\overline{*}^{[m]}\, y = s^m x = x$ for any $x, y \in X$. Hence we have $s^m = t^m = 1$, that is, $x *^m y = s^{-m}t^m x + (1 - s^{-m}t^m)y = x$ for any $x, y \in \mathcal{Q}(X)$. Therefore we have $m' \leq m$. We assume that $m = m'l_1 + l_2$ for some $l_1, l_2 \in \mathbb{Z}_{\geq 0}$ such that $0 < l_2 < m'$. Then we have $x *^m y = s^{-l_2}t^{l_2}x + (1 - s^{-l_2}t^{l_2})y = x$, which contradicts to $m' = \operatorname{type} \mathcal{Q}(X)$. Therefore we obtain $m = m'l_1$ for some $l_1 \in \mathbb{Z}_{\geq 0}$. $\square$

Here we see two examples. Let X be the Alexander biquandle $\mathbb{Z}[s^{\pm 1}, t^{\pm 1}]/(s - t)$. Then we have $\operatorname{type} X = \infty$ and $\operatorname{type} \mathcal{Q}(X) = 1$. Next, let X be the Alexander biquandle $\mathbb{Z}[s^{\pm 1}, t^{\pm 1}]/(s + t, t^4 - 1)$. Then we have $\operatorname{type} X = 4$ and $\operatorname{type} \mathcal{Q}(X) = 2$.

7.2 A relationship between MCQ/MCB colorings

In this section, we define a functor $\mathcal{Q}$ from the category of MCBs to that of MCQs, where we note that we use the same symbol $\mathcal{Q}$ as the above functor from the category of biquandles to that of quandles. We prove that for any MCB X, there is a one-to-one correspondence between the set of X-colorings and that of $\mathcal{Q}(X)$-colorings for any S^1-oriented handlebody-link.

We denote by MCQ (resp. MCB) the category of MCQs (resp. MCBs), whose objects are MCQs (resp. MCBs) and whose morphisms are MCQ homomorphisms (resp. MCB homomorphisms).

Definition 7.2.1. We define a functor $\mathcal{Q}$ from MCB to MCQ by $\mathcal{Q}((X, \underline{*}, \overline{*})) = (X, *)$ with $x * y = x \,\underline{*}\, y \,\overline{*}^{-1}\, y$ for any MCB $(X, \underline{*}, \overline{*})$ and $\mathcal{Q}(\phi) = \phi$ for any MCB homomorphism ϕ.

In the following, we see that the functor $\mathcal{Q}$ is well-defined.

Proposition 7.2.2. *The functor* $\mathcal{Q} : \mathsf{MCB} \to \mathsf{MCQ}$ *is well-defined.*

Proof. Let $X = \bigsqcup_{\lambda \in \Lambda} G_\lambda$ be an MCB. At first, since $a^{-1}b \,\overline{*}\, a = ba^{-1} \,\underline{*}\, a$ for any $a, b \in G_\lambda$, we have $a * b = a \,\underline{*}\, b \,\overline{*}^{-1}\, b = b^{-1}ab$. Second, for any $x \in X$ and $a, b \in G_\lambda$, $x \,\overline{*}^{-1}\, ab = x \,\overline{*}^{-1}\, (a \,\underline{*}\, b) \,\overline{*}^{-1}\, b$ since

$$\begin{aligned} x \,\overline{*}^{-1}\, (a \,\underline{*}\, b) \,\overline{*}^{-1}\, b \,\overline{*}\, ab &= x \,\overline{*}^{-1}\, (a \,\underline{*}\, b) \,\overline{*}^{-1}\, b \,\overline{*}\, b(a \,\underline{*}\, b \,\overline{*}^{-1}\, b) \\ &= (x \,\overline{*}^{-1}\, (a \,\underline{*}\, b) \,\overline{*}^{-1}\, b \,\overline{*}\, b) \,\overline{*}\, (a \,\underline{*}\, b \,\overline{*}^{-1}\, b \,\overline{*}\, b) \\ &= x \,\overline{*}^{-1}\, (a \,\underline{*}\, b) \,\overline{*}\, (a \,\underline{*}\, b) \\ &= x. \end{aligned}$$

Hence we have

$$\begin{aligned} x * ab &= x \,\underline{*}\, ab \,\overline{*}^{-1}\, ab \\ &= (x \,\underline{*}\, a) \,\underline{*}\, (b \,\overline{*}\, a) \,\overline{*}^{-1}\, (a \,\underline{*}\, b) \,\overline{*}^{-1}\, b \\ &= ((x \,\underline{*}\, a \,\overline{*}^{-1}\, a) \,\overline{*}\, a) \,\underline{*}\, (b \,\overline{*}\, a) \,\overline{*}^{-1}\, (a \,\underline{*}\, b) \,\overline{*}^{-1}\, b \\ &= ((x \,\underline{*}\, a \,\overline{*}^{-1}\, a) \,\underline{*}\, b) \,\overline{*}\, (a \,\underline{*}\, b) \,\overline{*}^{-1}\, (a \,\underline{*}\, b) \,\overline{*}^{-1}\, b \\ &= x \,\underline{*}\, a \,\overline{*}^{-1}\, a \,\underline{*}\, b \,\overline{*}^{-1}\, b \\ &= (x * a) * b. \end{aligned}$$

Furthermore, we can easily check that $x * e_\lambda = x$. Third, for any $x, y, z \in X$, we obtain $(x * y) * z = (x * z) * (y * z)$ since $\mathcal{Q}(X)$ is a quandle [2]. Finally, for any $x \in X$ and $a, b \in G_\lambda$,

$$\begin{aligned} ab * x &= ab \,\underline{*}\, x \,\overline{*}^{-1}\, x \\ &= (a \,\underline{*}\, x \,\overline{*}^{-1}\, x)(b \,\underline{*}\, x \,\overline{*}^{-1}\, x) \\ &= (a * x)(b * x) \end{aligned}$$

since $\underline{*}x : G_a \to G_{a \underline{*} x}$ and $\overline{*}x : G_a \to G_{a \overline{*} x}$ are group homomorphisms. Therefore $\mathcal{Q}(X)$ is an MCQ.

On the other hand, for any MCB homomorphism $\phi : X \to Y$ and $x, y \in X$, we have

$$\begin{aligned} \mathcal{Q}(\phi)(x * y) &= \phi(x * y) \\ &= \phi(x \,\underline{*}\, y \,\overline{*}^{-1}\, y) \\ &= \phi(x) \,\underline{*}\, \phi(y) \,\overline{*}^{-1}\, \phi(y) \\ &= \phi(x) * \phi(y) \\ &= \mathcal{Q}(\phi)(x) * \mathcal{Q}(\phi)(y). \end{aligned}$$

Hence $\mathcal{Q}(\phi)$ is an MCQ homomorphism from $\mathcal{Q}(X)$ to $\mathcal{Q}(Y)$. Furthermore it is clear that $\mathcal{Q}(\mathrm{id}_X) = \mathrm{id}_{\mathcal{Q}(X)}$ and $\mathcal{Q}(\psi \circ \phi) = \mathcal{Q}(\psi) \circ \mathcal{Q}(\phi)$. This completes the proof. $\square$

For an S^1-oriented handlebody-link H, the *reverse* of H, denoted $-H$, is obtained by reversing the orientations of all genus 1 components, and the *reflection* of H, denoted H^*, is the image of H under an orientation-reversing self-homeomorphism of S^3. A *split handlebody-link* is a handlebody-link whose exterior is reducible. For any handlebody-links H_1 and H_2, we denote by $H_1 \sqcup H_2$ the split handlebody-link H such that there exists a 2-sphere in $S^3 - H$ separating S^3 into two 3-balls, each of which contains only H_1 and H_2 respectively.

Let D and D' be diagrams of S^1-oriented handlebody-links H and H' respectively. In the following, we define diagrams $-D, D^v, D^h, D \sqcup D'$ and $W(D)$ (see Figure 7.1). We denote by $-D$ and D^v the diagrams of $-H$ and H^* obtained from D by reversing the orientations of all (semi)-arcs and switching all crossings respectively. We can regard that D is depicted in an xy-plane. Let ι be the involution $(x, y) \mapsto (-x, y)$. Then we define the diagram D^h of H^* by $D^h = \iota(D)$. We regard ι as the map from $\mathcal{A}(D)$ to $\mathcal{A}(D^h)$ (or $\mathcal{SA}(D)$ to $\mathcal{SA}(D^h)$).

An S^1-oriented handlebody-link diagram in S^2 is a *split diagram* if there is a loop in the exterior of the diagram separating S^2 into two disks each containing part of it. We denote by $D \sqcup D'$ the split diagram of $H \sqcup H'$ such that D and D' represent H and H' respectively. We denote by $W(D)$ the diagram of the S^1-oriented handlebody-link $H \sqcup -H^*$ obtained from $D \sqcup -D^v$ by sliding $-D^v$ under D and shifting it slightly to the normal orientations of all (semi-)arcs of D.

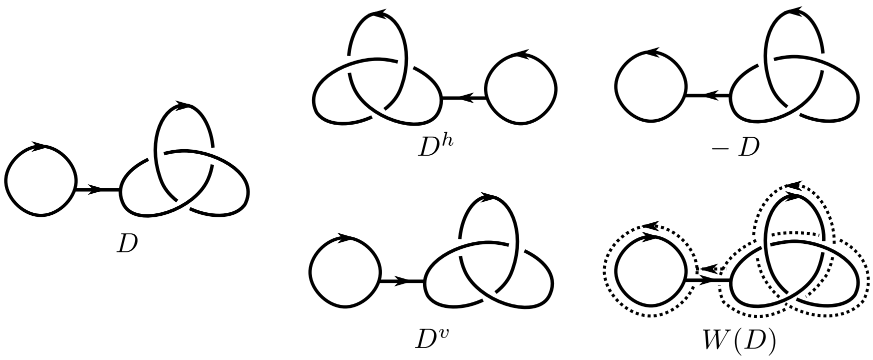


Figure 7.1: Diagrams $D, -D, D^v, D^h$ and $W(D)$.

Let X be an MCB. We note here that $\mathcal{SA}(-D) = \mathcal{SA}(D)$. For any $C \in \mathrm{Col}_X(D)$, we define $C^* \in \mathrm{Col}_X(-D^h)$ by $C^* = C \circ \iota$ as shown in Figure 7.2, where each x_i is an element of X. We note that the X-coloring C^* is shown in Figure 7.3 at each crossing and vertex. We define $C \sqcup C^* \in \mathrm{Col}_X(D \sqcup -D^h)$ by $(C \sqcup C^*)|_{\mathcal{SA}(D)} = C$ and $(C \sqcup C^*)|_{\mathcal{SA}(-D^h)} = C^*$. We set $\mathrm{Col}_X^{\sqcup}(D \sqcup -D^h) := \{C \sqcup C^* \mid C \in \mathrm{Col}_X(D)\}$. We denote by $\mathrm{Col}_X^{W}(W(D))$ the set of X-colorings of $W(D)$ satisfying the conditions depicted in Figure 7.4 at each crossing and vertex.

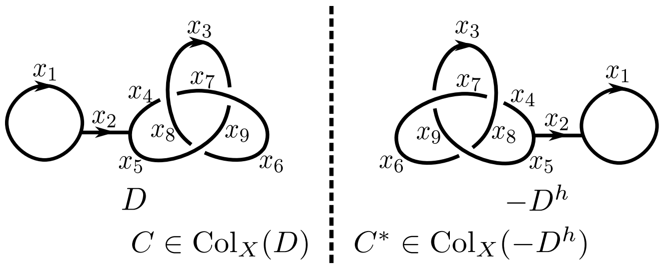


Figure 7.2: X-colorings C and C^*.

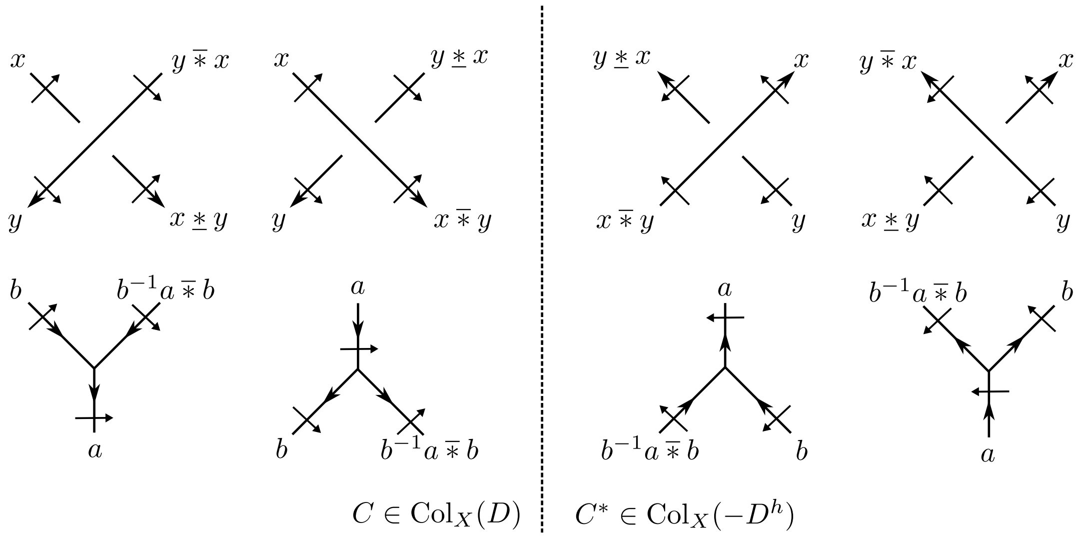


Figure 7.3: The well-definedness of $C^* \in \mathrm{Col}_X(-D^h)$.

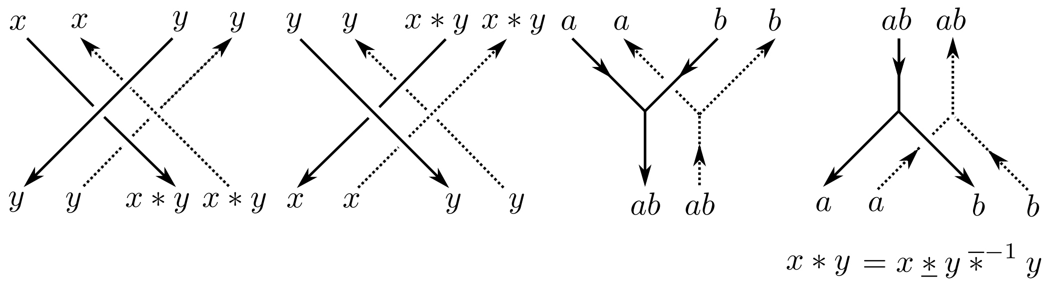


Figure 7.4: The coloring conditions of $\mathrm{Col}_X^W(W(D))$.

Lemma 7.2.3. *Let X be an MCB. For the X-coloring depicted in Figure 7.5, where x_i, x_i', y_i, y_i', z_i, z_i', w_i and w_i' are elements of X for any i, it follows that $(x_1, \ldots, x_l) = (x_1', \ldots, x_l')$ if and only if $(y_1, \ldots, y_l) = (y_1', \ldots, y_l')$.*

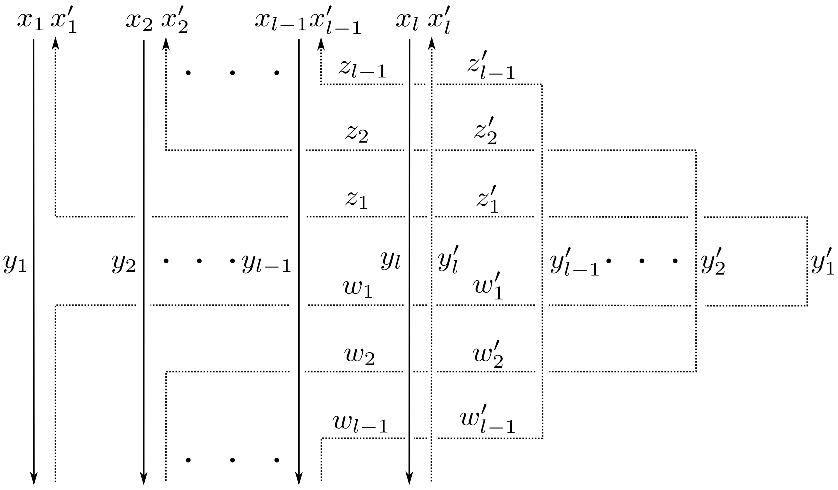


Figure 7.5:

Proof. We give the proof by induction on l. When $l = 1$, the statement holds immediately. Assume that the statement is proved for $l - 1$. Suppose that $(x_1, \ldots, x_l) = (x'_1, \ldots, x'_l)$. Then we have $z_i = z'_i$, $y_l = y'_l$ and $w_i = w'_i$ for any $i = 1, \ldots, l-1$ (see Figure 7.5). Hence we obtain the X-coloring depicted in Figure 7.6 from the X-coloring depicted in Figure 7.5. Therefore we have $(y_1, \ldots, y_{l-1}) = (y'_1, \ldots, y'_{l-1})$ by the assumption. Consequently, it follows that $(y_1, \ldots, y_l) = (y'_1, \ldots, y'_l)$. In the same way, if we suppose that $(y_1, \ldots, y_l) = (y'_1, \ldots, y'_l)$, then it follows that $(x_1, \ldots, x_l) = (x'_1, \ldots, x'_l)$, where we also have $z_i = z'_i$ and $w_i = w'_i$ for any $i = 1, \ldots, l-1$. $\square$

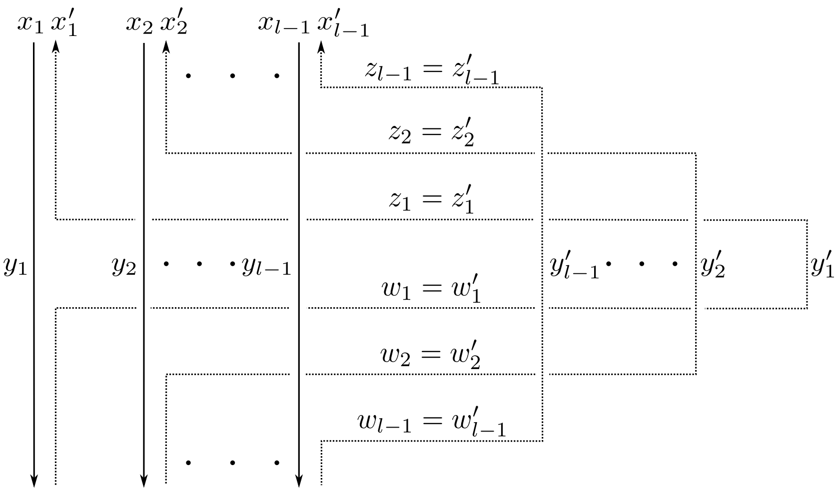


Figure 7.6:

Theorem 7.2.4. *Let X be an MCB and let D be a diagram of an S^1-oriented handlebody-link. Then there is a one-to-one correspondence between* $\mathrm{Col}_X(D)$ *and* $\mathrm{Col}_{\mathcal{Q}(X)}(D)$.

Proof. By [40], any S^1-oriented handlebody-link can be represented by $\mathrm{cl}(\mathrm{bind}^{m_1,\dots,m_s}_{n_1,\dots,n_s}(b_0))$, where b_0 is a classical l-braid diagram and $m_i, n_i \in \mathbb{Z}_{>0}$, and we can deform it into $\mathrm{cl}(b)$, where b is the trivalent braid diagram as shown in Figure 7.7. Then we may assume that D has the resulting form $\mathrm{cl}(b)$. Here we note that any MCQ(MCB)-coloring of D is determined by the colors of all (semi-)arcs incident to the top endpoints of the trivalent braid diagram b.

First, for any $C_1 \in \mathrm{Col}_{\mathcal{Q}(X)}(D)$, we denote by $\psi_1^X(C_1)$ the X-coloring of $W(D)$ depicted in Figure 7.8. Then ψ_1^X is a bijective map from $\mathrm{Col}_{\mathcal{Q}(X)}(D)$ to $\mathrm{Col}^W_X(W(D))$. Second, we can deform $W(D)$ into $D \sqcup -D^h$ by Reidemeister moves as shown in Figure 7.9. By Proposition 3.4.1 and Lemma 7.2.3, we obtain a bijective map ψ_2^X from $\mathrm{Col}^W_X(W(D))$ to $\mathrm{Col}^{\sqcup}_X(D \sqcup -D^h)$ as shown in Figure 7.9, where x_i and y_i are elements in X. Finally, we define a map ψ_3^X from $\mathrm{Col}^{\sqcup}_X(D \sqcup -D^h)$ to $\mathrm{Col}_X(D)$ by $\psi_3^X(C_3 \sqcup C_3^*) = C_3$, which is bijective obviously. Therefore $\psi_3^X \circ \psi_2^X \circ \psi_1^X$ is a bijective map from $\mathrm{Col}_{\mathcal{Q}(X)}(D)$ to $\mathrm{Col}_X(D)$. $\square$

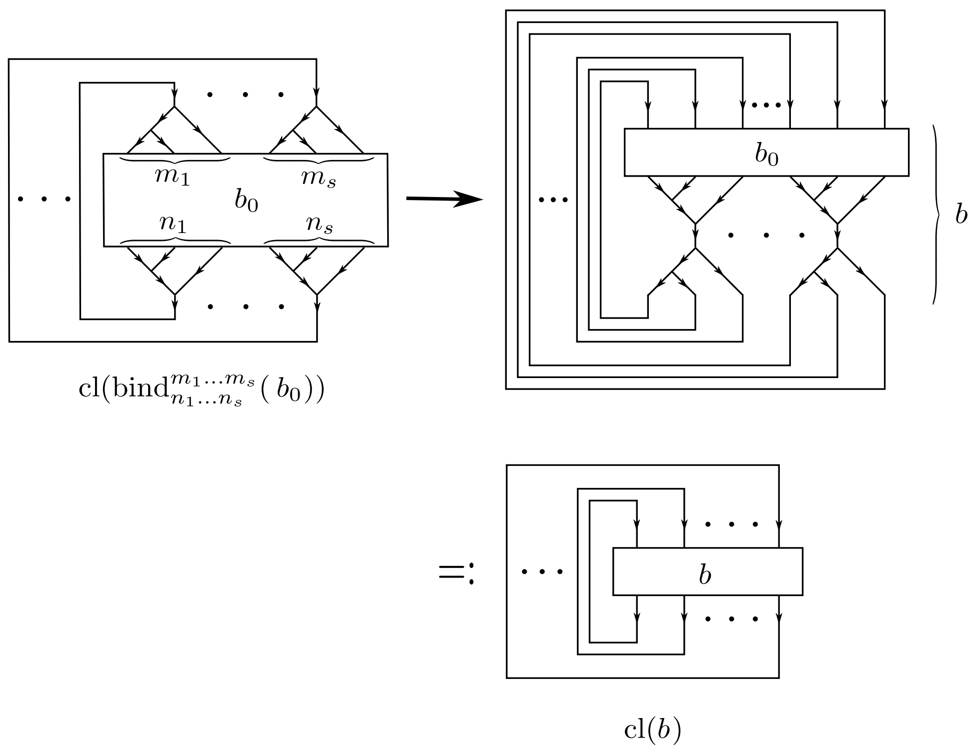


Figure 7.7: A closed trivalent braid diagram.

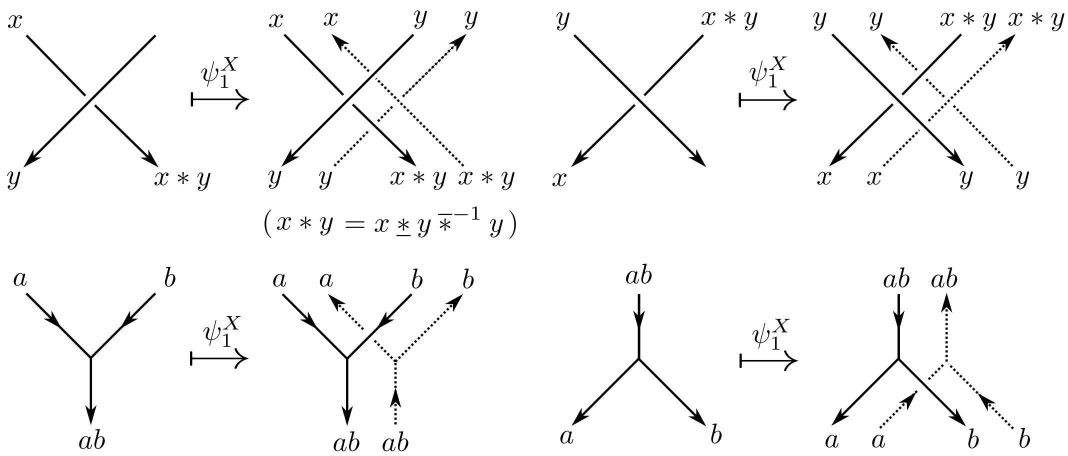


Figure 7.8: The map $\psi_1^X : \mathrm{Col}_{\mathcal{Q}(X)}(D) \to \mathrm{Col}_X^W(W(D))$.

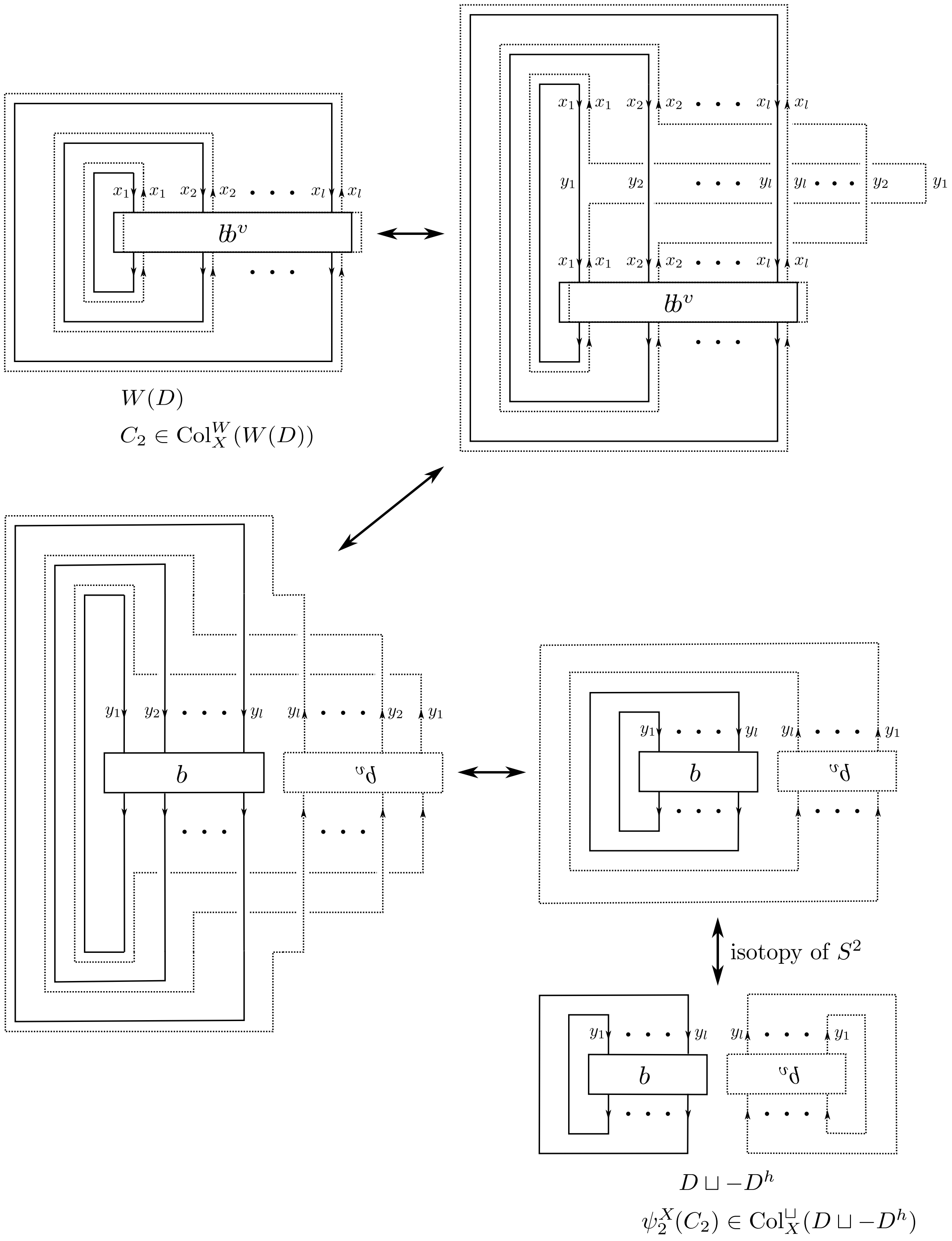


Figure 7.9: The deformation from $W(D)$ to $D \sqcup -D^h$.

7.3 A relationship between G-family of quandles/biquandles colorings

In this section, we show that there is the similar correspondence between G-family of quandles and G-family of biquandles colorings to between MCQ and MCB colorings.

Let $(X, \{\underline{\ast}^g\}_{g\in G}, \{\overline{\ast}^g\}_{g\in G})$ be a G-family of biquandles. For any $x, y \in X$ and $g \in G$, it follows that

$$(x \mathbin{\underline{\ast}^g} y) \mathbin{\underline{\ast}^{g^{-1}}} (y \mathbin{\underline{\ast}^g} y) = x \mathbin{\underline{\ast}^e} y = x$$

and

$$\begin{aligned} x \mathbin{\underline{\ast}^{g^{-1}}} (y \mathbin{\underline{\ast}^g} y) \mathbin{\underline{\ast}^g} y &= \{x \mathbin{\underline{\ast}^{g^{-1}}} (y \mathbin{\underline{\ast}^g} y)\} \mathbin{\underline{\ast}^g} \{(y \mathbin{\underline{\ast}^g} y) \mathbin{\underline{\ast}^{g^{-1}}} (y \mathbin{\underline{\ast}^g} y)\} \\ &= x \mathbin{\underline{\ast}^e} (y \mathbin{\underline{\ast}^g} y) \\ &= x. \end{aligned}$$

Hence the map $\underline{\ast}^g y : X \to X$, which sends x into $x \underline{\ast}^g y$, is a bijection and $(\underline{\ast}^g y)^{-1}(x) = x \underline{\ast}^{g^{-1}} (y \underline{\ast}^g y)$. Similarly, the map $\overline{\ast}^g y : X \to X$, which sends x into $x \overline{\ast}^g y$, is a bijection and $(\overline{\ast}^g y)^{-1}(x) = x \mathbin{\overline{\ast}^{g^{-1}}} (y \mathbin{\overline{\ast}^g} y)$. Then we have the following proposition.

Proposition 7.3.1. *Let $(X, \{\underline{\ast}^g\}_{g\in G}, \{\overline{\ast}^g\}_{g\in G})$ be a G-family of biquandles. Then $(X, \{\ast^g\}_{g\in G})$ is a G-family of quandles by defining $x \ast^g y = (x \mathbin{\underline{\ast}^g} y) \mathbin{\overline{\ast}^{g^{-1}}} (y \mathbin{\overline{\ast}^g} y)$.*

Proof.

- For any $x \in X$ and $g \in G$,

$$x \ast^g x = (x \mathbin{\underline{\ast}^g} x) \mathbin{\overline{\ast}^{g^{-1}}} (x \mathbin{\overline{\ast}^g} x) = (x \mathbin{\overline{\ast}^g} x) \mathbin{\overline{\ast}^{g^{-1}}} (x \mathbin{\overline{\ast}^g} x) = x \mathbin{\overline{\ast}^e} x = x.$$

- For any $x, y \in X$, $g, h \in G$,

$$\begin{aligned} & x \ast^{gh} y \mathbin{\overline{\ast}^g} y \mathbin{\overline{\ast}^h} (y \mathbin{\overline{\ast}^g} y) \\ &= x \mathbin{\underline{\ast}^{gh}} y \mathbin{\overline{\ast}^{h^{-1}g^{-1}}} (y \mathbin{\overline{\ast}^{gh}} y) \mathbin{\overline{\ast}^g} y \mathbin{\overline{\ast}^h} (y \mathbin{\overline{\ast}^g} y) \\ &= x \mathbin{\underline{\ast}^{gh}} y \mathbin{\overline{\ast}^{h^{-1}}} (y \mathbin{\overline{\ast}^{gh}} y) \mathbin{\overline{\ast}^{g^{-1}}} \{(y \mathbin{\overline{\ast}^{gh}} y) \mathbin{\overline{\ast}^{h^{-1}}} (y \mathbin{\overline{\ast}^{gh}} y)\} \mathbin{\overline{\ast}^g} y \mathbin{\overline{\ast}^h} (y \mathbin{\overline{\ast}^g} y) \\ &= x \mathbin{\underline{\ast}^{gh}} y \mathbin{\overline{\ast}^{h^{-1}}} (y \mathbin{\overline{\ast}^{gh}} y) \mathbin{\overline{\ast}^{g^{-1}}} (y \mathbin{\overline{\ast}^g} y) \mathbin{\overline{\ast}^g} y \mathbin{\overline{\ast}^h} (y \mathbin{\overline{\ast}^g} y) \\ &= x \mathbin{\underline{\ast}^{gh}} y \mathbin{\overline{\ast}^{h^{-1}}} (y \mathbin{\overline{\ast}^{gh}} y) \mathbin{\overline{\ast}^h} (y \mathbin{\overline{\ast}^g} y) \\ &= x \mathbin{\underline{\ast}^{gh}} y \mathbin{\overline{\ast}^{h^{-1}}} \{(y \mathbin{\overline{\ast}^g} y) \mathbin{\overline{\ast}^h} (y \mathbin{\overline{\ast}^g} y)\} \mathbin{\overline{\ast}^h} (y \mathbin{\overline{\ast}^g} y) \\ &= x \mathbin{\underline{\ast}^{gh}} y. \end{aligned}$$

  On the other hand,

$$\begin{aligned} & (x \ast^g y) \ast^h y \mathbin{\overline{\ast}^g} y \mathbin{\overline{\ast}^h} (y \mathbin{\overline{\ast}^g} y) \\ &= \{x \ast^g y \mathbin{\underline{\ast}^h} y \mathbin{\overline{\ast}^{h^{-1}}} (y \mathbin{\overline{\ast}^h} y) \mathbin{\overline{\ast}^g} y\} \mathbin{\overline{\ast}^h} (y \mathbin{\overline{\ast}^g} y) \\ &= \{x \ast^g y \mathbin{\underline{\ast}^h} y \mathbin{\overline{\ast}^{h^{-1}}} (y \mathbin{\overline{\ast}^h} y) \mathbin{\overline{\ast}^h} y\} \mathbin{\overline{\ast}^{h^{-1}gh}} (y \mathbin{\underline{\ast}^h} y) \\ &= (x \ast^g y \mathbin{\underline{\ast}^h} y) \mathbin{\overline{\ast}^{h^{-1}gh}} (y \mathbin{\underline{\ast}^h} y) \\ &= \{(x \mathbin{\underline{\ast}^g} y) \mathbin{\overline{\ast}^{g^{-1}}} (y \mathbin{\overline{\ast}^g} y) \mathbin{\overline{\ast}^g} y\} \mathbin{\underline{\ast}^h} (y \mathbin{\overline{\ast}^g} y) \\ &= (x \mathbin{\underline{\ast}^g} y) \mathbin{\underline{\ast}^h} (y \mathbin{\underline{\ast}^g} y) \\ &= x \mathbin{\underline{\ast}^{gh}} y. \end{aligned}$$

Therefore we have $x *^{gh} y = (x *^g y) *^h y$. we can easily check that $x *^e y = x$ for any $x, y \in X$.

- For any $x, y, z \in X$, $g, h \in G$ and $\alpha = (y \underline{*}^h z) \overline{*}^{h^{-1}} (z \overline{*}^h z)$,

$$
\begin{aligned}
&(x *^g y) *^h z \overline{*}^{h^{-1}gh} \alpha \overline{*}^{h^{-1}g^{-1}hgh} (z \underline{*}^{h^{-1}gh} \alpha)\\
&= \{((x \underline{*}^g y) \overline{*}^{g^{-1}} (y \overline{*}^g y) \underline{*}^h z \overline{*}^{h^{-1}} (z \overline{*}^h z)) \overline{*}^{h^{-1}gh} \alpha\} \overline{*}^{h^{-1}g^{-1}hgh} (z \underline{*}^{h^{-1}gh} \alpha)\\
&= \{((x \underline{*}^g y) \overline{*}^{g^{-1}} (y \overline{*}^g y) \underline{*}^h z \overline{*}^{h^{-1}} (z \overline{*}^h z)) \overline{*}^h z\} \overline{*}^{h^{-1}gh} (\alpha \overline{*}^h z)\\
&= \{(x \underline{*}^g y) \overline{*}^{g^{-1}} (y \overline{*}^g y) \underline{*}^h z\} \overline{*}^{h^{-1}gh} (y \underline{*}^h z)\\
&= \{(x \underline{*}^g y) \overline{*}^{g^{-1}} (y \overline{*}^g y) \overline{*}^g y\} \underline{*}^h (z \overline{*}^g y)\\
&= (x \underline{*}^g y) \underline{*}^h (z \overline{*}^g y).
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
&(x *^h z) *^{h^{-1}gh} (y *^h z) \overline{*}^{h^{-1}gh} \alpha \overline{*}^{h^{-1}g^{-1}hgh} (z \underline{*}^{h^{-1}gh} \alpha)\\
&= ((x \underline{*}^h z) \overline{*}^{h^{-1}} (z \overline{*}^h z)) *^{h^{-1}gh} ((y \underline{*}^h z) \overline{*}^{h^{-1}} (z \overline{*}^h z)) \overline{*}^{h^{-1}gh} \alpha\\
&\quad \overline{*}^{h^{-1}g^{-1}hgh} (z \underline{*}^{h^{-1}gh} \alpha)\\
&= ((x \underline{*}^h z) \overline{*}^{h^{-1}} (z \overline{*}^h z)) \underline{*}^{h^{-1}gh} \alpha \overline{*}^{h^{-1}g^{-1}h} (\alpha \overline{*}^{h^{-1}gh} \alpha) \overline{*}^{h^{-1}gh} \alpha\\
&\quad \overline{*}^{h^{-1}g^{-1}hgh} (z \underline{*}^{h^{-1}gh} \alpha)\\
&= \{((x \underline{*}^h z) \overline{*}^{h^{-1}} (z \overline{*}^h z)) \underline{*}^{h^{-1}gh} \alpha\} \overline{*}^{h^{-1}g^{-1}hgh} (z \underline{*}^{h^{-1}gh} \alpha)\\
&= \{((x \underline{*}^h z) \overline{*}^{h^{-1}} (z \overline{*}^h z)) \overline{*}^h z\} \underline{*}^{h^{-1}gh} \{((y \underline{*}^h z) \overline{*}^{h^{-1}} (z \overline{*}^h z)) \overline{*}^h z\}\\
&= (x \underline{*}^h z) \underline{*}^{h^{-1}gh} (y \underline{*}^h z)\\
&= (x \underline{*}^g y) \underline{*}^h (z \overline{*}^g y).
\end{aligned}
$$

Therefore we have $(x *^g y) *^h z = (x *^h z) *^{h^{-1}gh} (y *^h z)$.

$\square$

By Proposition 7.3.1, for any G-family of biquandles $(X, \{\underline{*}^g\}_{g \in G}, \{\overline{*}^g\}_{g \in G})$, we have a G-family of quandles $(X, \{*^g\}_{g \in G})$, denoted by $\mathcal{Q}_G(X)$, by defining $x *^g y = (x \underline{*}^g y) \overline{*}^{g^{-1}} (y \overline{*}^g y)$. Then $\mathcal{Q}_G$ is a map from the set of G-families of biquandles to that of G-families of quandles. In particular, let $(X, \{\underline{*}^g\}_{g \in G}, \{\overline{*}^g\}_{g \in G})$ be a G-family of Alexander biquandles, where X is a right $R[G]$-module for some ring R and group G with a homomorphism $f : G \to Z(G)$ and where $Z(G)$ is the center of G. Then $\mathcal{Q}_G(X)$ is a G-family of Alexander quandles with the action $xg := xgf(g)$ since for any $x, y \in X$ and $g \in G$, we have $x *^g y = (x \underline{*}^g y) \overline{*}^{g^{-1}} (y \overline{*}^g y) = xgf(g) + y(e - gf(g))$.

For any G-family of biquandles $(X, \{\underline{*}^g\}_{g \in G}, \{\overline{*}^g\}_{g \in G})$ and its associated MCB $X \times G$, the MCQ $\mathcal{Q}(X \times G)$ coincides with the associated MCQ $\mathcal{Q}_G(X) \times G$ of the G-family of quandles $\mathcal{Q}_G(X)$ with $(x, g) * (y, h) = ((x \underline{*}^g y) \overline{*}^{g^{-1}} (y \overline{*}^g y), h^{-1}gh)$.

We remind that when X is a G-family of Alexander (bi)quandles as a right $R[G]$-module for some ring R, for any G-flowed diagram (D, ρ) of an S^1-oriented handlebody-link, the coloring set $\mathrm{Col}_X(D, \rho)$ is a right R-module with the action $(C \cdot r)(\alpha) := C(\alpha)r$ and the addition $(C + C')(\alpha) := C(\alpha) + C'(\alpha)$ for any $C, C' \in \mathrm{Col}_X(D, \rho)$, $\alpha \in \mathcal{A}(D, \rho)$ (or $\alpha \in \mathcal{SA}(D, \rho)$) and $r \in R$.

Let X be a G-family of quandles (resp. biquandles), $X\times G$ be the associated MCQ (resp. MCB) of X and let pr_G and pr_X be the natural projections from $X \times G$ to G and from $X\times G$ to X respectively. For any $\rho \in \mathrm{Flow}(D;G)$, we define $\mathrm{Col}^{\rho}_{X\times G}(D) := \{C \in \mathrm{Col}_{X\times G}(D) \mid \mathrm{pr}_G \circ C = \rho\}$, where for any $\alpha \in \mathcal{SA}(D)$ and $\widetilde{\alpha} \in \mathcal{A}(D)$ satisfying $\alpha \subset \widetilde{\alpha}$, we put $\rho(\alpha) := \rho(\widetilde{\alpha})$ when X is a G-family of biquandles. Then we can identify $\mathrm{Col}^{\rho}_{X\times G}(D)$ with $\mathrm{Col}_X(D,\rho)$, that is, for any $C \in \mathrm{Col}^{\rho}_{X\times G}(D)$, the map $\mathrm{pr}_G \circ C$ corresponds to the G-flow ρ of D, and the map $\mathrm{pr}_X \circ C$ corresponds to the X-coloring of (D,ρ). Therefore $\mathrm{Col}^{\rho}_{X\times G}(D)$ is also a right R-module in the same way as $\mathrm{Col}_X(D,\rho)$. Then we obtain the following corollary by Theorem 7.2.4.

Corollary 7.3.2. *Let X be a G-family of biquandles and let (D,ρ) be a G-flowed diagram of an S^1-oriented handlebody-link. Then there is a one-to-one correspondence between $\mathrm{Col}_X(D,\rho)$ and $\mathrm{Col}_{\mathcal{Q}_G(X)}(D,\rho)$. In particular, when X is a G-family of Alexander biquandles, $\mathrm{Col}_X(D,\rho)$ is isomorphic to $\mathrm{Col}_{\mathcal{Q}_G(X)}(D,\rho)$ as right R-modules.*

Proof. We remind that we can identify $\mathrm{Col}^{\rho}_{X\times G}(D)$ with $\mathrm{Col}_X(D,\rho)$ and $\mathrm{Col}^{\rho}_{\mathcal{Q}_G(X)\times G}(D)$ with $\mathrm{Col}_{\mathcal{Q}_G(X)}(D,\rho)$, and we note that

$$\mathrm{Col}_{X\times G}(D) = \bigsqcup_{\rho'\in \mathrm{Flow}(D;G)} \mathrm{Col}^{\rho'}_{X\times G}(D)$$

and

$$\mathrm{Col}_{\mathcal{Q}(X\times G)}(D) = \mathrm{Col}_{\mathcal{Q}_G(X)\times G}(D) = \bigsqcup_{\rho'\in \mathrm{Flow}(D;G)} \mathrm{Col}^{\rho'}_{\mathcal{Q}_G(X)\times G}(D).$$

By the proof of Theorem 7.2.4, the map $\Psi^{X\times G} := \psi_3^{X\times G}\circ\psi_2^{X\times G}\circ\psi_1^{X\times G}$ is a bijective map from $\mathrm{Col}_{\mathcal{Q}_G(X)\times G}(D)$ to $\mathrm{Col}_{X\times G}(D)$, and $\Psi^{X\times G}(\mathrm{Col}^{\rho'}_{\mathcal{Q}_G(X)\times G}(D)) \subset \mathrm{Col}^{\rho'}_{X\times G}(D)$ for any $\rho' \in \mathrm{Flow}(D;G)$ (see Figures 7.8 and 7.9). Hence $\Psi^{X\times G}|_{\mathrm{Col}^{\rho}_{\mathcal{Q}_G(X)\times G}(D)}$ is a bijective map from $\mathrm{Col}^{\rho}_{\mathcal{Q}_G(X)\times G}(D)$ to $\mathrm{Col}^{\rho}_{X\times G}(D)$. Next, suppose that X is a G-family of Alexander biquandles. Then $\psi_1^{X\times G}$ and $\psi_3^{X\times G}$ preserve module structures clearly. Furthermore $\psi_2^{X\times G}$ also preserves module structures since in Lemma 7.2.3, each y_i can be represented by using each x_i and the operations $\underline{*}$ and $\overline{*}$. Therefore $\Psi^{X\times G}|_{\mathrm{Col}^{\rho}_{\mathcal{Q}_G(X)\times G}(D)}$ is an isomorphism of right R-modules. □

Finally, we see an example. Let (D_{A_n}, ρ_{A_n}) be the $\mathbb{Z}_8$-flowed diagram of the handlebody-knot A_n depicted in Figure 7.10 for any $n \in \mathbb{Z}_{>0}$. Let $s = t+1 \in \mathbb{Z}_3[t^{\pm 1}]$ and let $f(t) = t^2+t+2 \in \mathbb{Z}_3[t^{\pm1}]$, which is an irreducible polynomial. Then $X := \mathbb{Z}_3[t^{\pm1}]/(f(t))$ is a $\mathbb{Z}_8$-family of Alexander biquandles and a field. By Example 5.4.3, it follows that $\dim \mathrm{Col}_X(D_{A_n}, \rho_{A_n}) = n+1$ as vector spaces over X, and the assignment of elements $x_0,\dots,x_n$ of X to each semi-arc of (D_{A_n}, ρ_{A_n}) as shown in Figure 7.10 corresponds to a basis of $\mathrm{Col}_X(D_{A_n}, \rho_{A_n})$. By Proposition 7.3.1, $\mathcal{Q}_G(X)$ is a $\mathbb{Z}_8$-family of Alexander quandles with $x *^i y = s^{-i}t^i x + (1-s^{-i}t^i)y = t^{2i}x + (1-t^{2i})y$ for any $i \in \mathbb{Z}_8$. By Corollary 7.3.2, we have $\dim \mathrm{Col}_{\mathcal{Q}_G(X)}(D_{A_n}, \rho_{A_n}) = n+1$ as vector spaces over X, and the assignment of elements $x_0,\dots,x_n$ of X to each arc of (D_{A_n}, ρ_{A_n}) as shown in Figure 7.10 corresponds to a basis of $\mathrm{Col}_{\mathcal{Q}_G(X)}(D_{A_n}, \rho_{A_n})$.

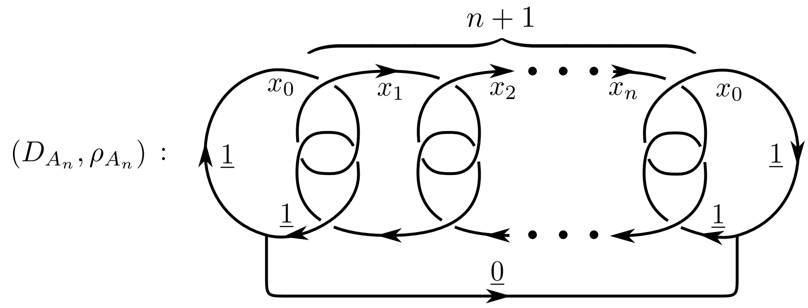


Figure 7.10: A $\mathbb{Z}_8$-flowed diagram (D_{A_n}, ρ_{A_n}) of A_n.

Acknowledgement

The author would like to express his best gratitude to his supervisor Atsushi Ishii for his constant encouragement and invaluable discussions. The author would like to thank Kazuhiro Kawamura for his warm encouragement since undergraduate years. The author also would like to express his thanks to Yusuke Iijima, Katsumi Ishikawa, Masahide Iwakiri, Shosaku Matsuzaki, Shin'ya Okazaki, Yusuke Suzuki and Kokoro Tanaka for their beneficial comments and suggestions. Finally, the author is grateful to his parents for their unconditional support.

References

[1] J. W. Alexander, *A lemma on systems of knotted curves*, Proc. Natl. Acad. Sci. USA **9**(1923), 93–95.

[2] S. Ashihara, *Fundamental biquandles of ribbon 2-knots and ribbon torus-knots with isomorphic fundamental quandles*, J. Knot Theory Ramifications **23**(2014), 1450001, 17 pp.

[3] S. Cho and D. McCullough, *Cabling sequences of tunnels of torus knots*, Algebr. Geom. Topol. **9**(2009), 1–20.

[4] S. Cho and D. McCullough, *The tree of knot tunnels*, Geom. Topol. **13**(2009), 769–815.

[5] S. Cho and D. McCullough, *Constructing knot tunnels using giant steps*, Proc. Amer. Math. Soc. **138**(2010), 375–384.

[6] S. Cho and D. McCullough, *Tunnel leveling, depth, and bridge numbers*, Trans. Amer. Math. Soc. **363**(2011), 259–280.

[7] W. E. Clark, M. Elhamdadi, M. Saito and T. Yeatman, *Quandle colorings of knots and applications*, J. Knot Theory Ramifications **23**(2014), 1450035, 23 pp.

[8] G. Ehrman, A. Gurpinar, M. Thibault and D. N. Yetter, *Toward a classification of finite quandles*, J. Knot Theory Ramifications **17**(2008), 511–520.

[9] R. Fenn, M. Jordan-Santana and L. H. Kauffman, *Biquandles and virtual links*, Topology Appl. **145**(2004), 157–175.

[10] R. Fenn, C. Rourke and B. Sanderson, *Trunks and classifying spaces*, Appl. Categ. Structure **3**(1995), 321–356.

[11] C. Gordon and J. Luecke, *Knots are determined by their complements*, J. Amer. Math. Soc. **2**(1989), 371–415.

[12] M. Graña, *Indecomposable racks of order p^2*, Beiträge Algebra Geom. **45**(2004), 665–767.

[13] Y. Iijima, ハンドル体絡み目とカンドルの連結性, Master's Thesis, University of Tsukuba (2017).

[14] A. Ishii, *Moves and invariants for knotted handlebodies*, Algebr. Geom. Topol. **8**(2008), 1403–1418.

[15] A. Ishii, *A multiple conjugation quandle and handlebody-knots*, Topology Appl. **196**(2015), 492–500.

[16] A. Ishii, *The Markov theorems for spatial graphs and handlebody-knots with Y-orientations*, Internat. J. Math. **26**(2015), 1550116, 23 pp.

[17] A. Ishii and M. Iwakiri, *Quandle cocycle invariants for spatial graphs and knotted handlebodies*, Canad. J. Math. **64**(2012), 102–122.

[18] A. Ishii, M. Iwakiri, Y. Jang and K. Oshiro, *A G-family of quandles and handlebody-knots*, Ill. J. Math. **57**(2013), 817–838.

[19] A. Ishii, M. Iwakiri, S. Kamada, J. Kim, S. Matsuzaki and K. Oshiro, *A multiple conjugation biquandle and handlebody-links*, Hiroshima Math. J. **48**(2018), no. 1, 89–117.

[20] A. Ishii and K. Kishimoto, *The IH-complex of spatial trivalent graphs*, Tokyo. J. Math. **33**(2010), 523–535.

[21] A. Ishii, K. Kishimoto, H. Moriuchi and M. Suzuki, *A table of genus two handlebody- knots up to six crossings*, J. Knot Theory Ramifications **21**(2012), 1250035, 1–9.

[22] A. Ishii, K. Kishimoto and M. Ozawa, *Knotted handle decomposing spheres for handlebody-knots*, J. Math. Soc. Japan **67**(2015), 407–417.

[23] A. Ishii and S. Nelson, *Partially multiplicative biquandles and handlebody-knots*, Contemp. Math. **689**(2017), 159–176.

[24] K. Ishikawa, *Knot quandles vs. knot biquandles*, preprint.

[25] K. Ishikawa and K. Tanaka, *Quandle coloring vs. biquandle coloring*, preprint.

[26] M. Iwakiri, *Unknotting numbers for handlebody-knots and Alexander quandle colorings*, J. Knot Theory Ramifications **24**(2015), 1550059, 13 pp.

[27] D. Joyce, *A classifying invariant of knots, the knot quandle*, J. Pure Appl. Alg. **23**(1982), 37–65.

[28] L. H. Kauffman, *Invariants of graphs in three-space*, Trans. Amer. Math. Soc. **311**(1989), 697–710.

[29] L. H. Kauffman, *Virtual knot theory*, Europ. J. Comb. **20**(1999), 663–691.

[30] L. H. Kauffman and D. E. Radford, *Bi-oriented quantum algebras, and generalized Alexander polynomial for virtual links*, Contemp. Math. **318**(2003), 113–140.

[31] J. Kim, *On critical Heegaard splittings of tunnel number two composite knot exteriors*, J. Knot Theory Ramifications **22**(2013), 1350065, 11 pp.

[32] J. H. Lee and S. Lee, *Inequivalent handlebody-knots with homeomorphic complements*, Algebr. Geom. Topol. **12**(2012), 1059–1079.

[33] J. Mandemaker, *Various topics in rack and quandle homology*, Master's Thesis, Radboud University (2010).

[34] S. V. Matveev, *Distributive groupoids in knot theory*, Mt. Sb. (N.S.) **119(161)**(1982), 78–88.

[35] Y. Moriah and H. Rubinstein, *Heegaard structures of negatively curved 3-manifolds*, Comm. in Anal. and Geom. **5**(1997), 375–412.

[36] K. Morimoto, *On the additivity of tunnel number of knots*, Topology Appl. **53**(1993), 37–66.

[37] K. Morimoto, *On the super additivity of tunnel number of knots*, Math. Ann. **317**(2000), 489–508.

[38] K. Morimoto, *On Heegaard splittings of knot exteriors with tunnel number degenerations*, Topology Appl. **196**(2015), 719–728.

[39] M. Motto, *Inequivalent genus 2 handlebodies in S^3 with homeomorphic complement*, Topology Appl. **36**(1990), 283–290.

[40] T. Murao, *On bind maps for braids*, J. Knot Theory Ramifications **25**(2016), 1650004, 25 pp.

[41] Y. Nakanishi, *A note on unknotting number*, Math. Sem. Notes, Kobe Univ. **9**(1981), 99–108.

[42] S. Nelson, *Classification of finite Alexander quandles*, Topology Proceedings **27**(2003), 245–258.

[43] S. Nelson and G. Murillo, *Alexander quandles of order 16*, J. Knot Theory Ramifications **17**(2008), 273–278.

[44] S. Nelson and C.-Y. Wong, *On the orbit decomposition of finite quandles*, J. Knot Theory Ramifications **15**(2006), 761–772.

[45] J. Przytycki, *3-coloring and other elementary invariants of knots*, Banach Center Publ. **42**(1998), 275–295.

[46] D. Rolfsen, *Knots and Links*, Math. Lecture Series no. 7, Publish or Perish Inc., Berkeley (1976).

[47] M. Scharlemann and J. Schultens, *The tunnel number of the sum of n knots is at least n*, Topology **38**(1999), 265–270.

[48] A. Soloviev, *Non-unitary set-theoretical solutions to the quantum Yang-Baxter equation*, Math. Res. Lett. **7(5-6)**(2000), 577–596.

[49] S. Yamada, *An invariant of spatial graphs*, J. Graph Theory **13**(1989), 537–551.

[50] G. Yang and F. Lei, *Some sufficient conditions for tunnel numbers of connected sum of two knots not to go down*, Acta Math. Sin. (Engl. Ser.) **27**(2011), 2229–2244.

[51] D. N. Yetter, *Category theoretic representations of knotted graphs in S^3*, Adv. Math. **77**(1989), 137–255.